Euler-Cauchy Using Undetermined Coefficients

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Outline

1. Introduction
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3. Second Order Euler-Cauchy with Monomial Right-Hand Side
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   - Case 2: $\alpha$ is a root of multiplicity one
   - Case 3: $\alpha$ is a double root
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In most differential equations courses, the homogeneous second order Euler-Cauchy equation,

\[ t^2 y'' + aty' + by = 0, \quad t \neq 0, \tag{1} \]

is one of the first higher order differential equations (DEs) with variable coefficients students see.

Some students (to my surprise) applied undetermined coefficients to directly solve certain exam problems involving nonhomogeneous Euler-Cauchy equations.

Questions:
1. Can we find a particular solution to this equation using substitution similar to standard undetermined coefficients?
2. If so, when?
The form of (1) leads us to seek solutions of the form
\[ y(t) = t^\lambda, \] where \( \lambda \) is a constant to be determined.

Plugging this into (1), gives the characteristic equation:
\[ \lambda^2 + (a - 1)\lambda + b = 0, \] to be solved for \( \lambda \).

Result:
- If \((a - 1)^2 > 4b\), \( y(t) = c_1 |t|^\lambda_1 + c_2 |t|^\lambda_2. \)
- If \((a - 1)^2 = 4b\), \( y(t) = c_1 |t|^\lambda + c_2 |t|^{\lambda \ln |t|}. \)
- If \((a - 1)^2 < 4b\), let \( \lambda_{1,2} = \alpha \pm i\beta; \) then
  \[ y(t) = |t|^{\alpha} (c_1 \cos(\beta \ln |t|) + c_2 \sin(\beta \ln |t|)). \]
Review: Undetermined Coefficients

- Always applicable only to constant-coefficient DEs with certain right-hand side functions.
- Idea: guess the form of the particular solution $y_p$ based on the type of right-hand side function. For example:
  - for an exponential, $a e^{kt}$, guess $y_p = A e^{kt}$;
  - for a polynomial (or monomial) of degree $n$, guess $y_p = C_0 + C_1 t + \ldots + C_n t^n$ (a polynomial of the same degree).

- Multiply $y_p$ by $t$ until it contains no part of the complementary solution.
- Plug $y_p$ into the DE and solve for the constant(s).
Assume that our Euler-Cauchy equation is given as
\[ t^2 y'' + aty' + by = f(t), \quad t > 0. \]

Change of variables: define \( t = e^z \).

Result:
\[
\frac{d^2 y}{dz^2} + (a - 1) \frac{dy}{dz} + by = f(e^z),
\]
a constant-coefficient DE.

Thus, if \( f(e^z) \) is one of the “special” right-hand side functions, can apply undetermined coefficients to the transformed DE.

Leads to a method of undetermined coefficients for the original equation.
Consider the second order Euler-Cauchy equation with a monomial right-hand side function:

\[ t^2 y'' + aty' + by = At^\alpha, \quad t > 0, \quad (2) \]

where \( \alpha \) is a real number.

- Three possibilities:
  - Case 1: \( \alpha \) is not a root of the characteristic equation,
  - Case 2: \( \alpha \) is a root of multiplicity one, or
  - Case 3: \( \alpha \) is a double root.
Case 1: \( \alpha \) is not a root of the characteristic equation

- Try as our particular solution a monomial of degree \( \alpha \), 
  \[ y_p(t) = Ct^\alpha. \]
- \( y_p \) contains no solution of the complementary equation, so keep going.
- Plug \( y_p \) into (2):
  \[ (\alpha(\alpha - 1) + a\alpha + b)Ct^\alpha = At^\alpha. \]
- Since \( \alpha \) is not a root of the characteristic equation and \( t \neq 0 \), obtain a unique solution for \( C \).
Case 2: $\alpha$ is a root of multiplicity one

- Recall: the Euler-Cauchy equation can be transformed into a constant-coefficient equation by the change of variables $t = e^z$.
- First guess for the particular solution of the transformed equation would be $y_p(z) = Ce^{\alpha z}$.
- Since $\alpha$ is a root of the characteristic equation, we need to multiply by $z$.
- Translates into multiplication by $\ln(t)$ in the particular solution for (2), so $y_p(t) = Ct^\alpha(\ln(t))$. 

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Case 3: $\alpha$ is a double root

- Similar to Case 2: look at the constant-coefficient equation.
- First guess for the particular solution of the transformed equation would be $y_p(z) = C e^{\alpha z}$.
- Since $\alpha$ is a double root of the characteristic equation, we need to multiply by $z^2$.
- Translates into multiplication by $(\ln(t))^2$ in the particular solution for (2), so $y_p(t) = C t^\alpha (\ln(t))^2$. 
Case 1: $\alpha$ is not a root of the characteristic equation

Case 2: $\alpha$ is a root of multiplicity one

Case 3: $\alpha$ is a double root

Summary

Theorem

For the second order Euler-Cauchy problem,

$$t^2 y'' + aty' + by = At^\alpha, \ t > 0,$$

where $\alpha \in \mathbb{R}$, a particular solution is of the form

(i) $y_p(t) = Ct^\alpha$, provided that $\alpha$ is not equal to any root of the characteristic equation, or

(ii) $y_p(t) = Ct^\alpha(\ln(t))^i$, if $\alpha$ is equal to a root of the characteristic equation, where $i$ is the multiplicity of the root.
Example: Find a general solution of 
\[ t^2 y'' - 4ty' + 4y = 4t^3 - 2t, \quad t > 0. \]

- Complementary solution: \( y_c = c_1 t + c_2 t^4. \)
- Particular solution: Solve \( t^2 y'' - 4ty' + 4y = 4t^3 - 2t. \)
  - Use superposition to apply Theorem 1 to each part of right-hand side.
  - Guess for \( y_p: y_p = At^3 + Bt \ln(t). \)
  - Plug in and collect terms: \(-2At^3 - 3Bt = 4t^3 - 2t.\)
  - Result: \( y_p = -2t^3 + \frac{2}{3} t \ln(t). \)
- General solution: \( y = y_c + y_p, \) so 
  \[
y(t) = c_1 t + c_2 t^4 - 2t^3 + \frac{2}{3} t \ln(t). \]
Example: Find a general solution of \( t^2 y'' - 4ty' + 4y = 4t^3 - 2t, \ t > 0. \)

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  \[
y(t) = c_1 t + c_2 t^4 - 2t^3 + \frac{2}{3} t \ln(t).
  \]
Can apply above approach to Euler-Cauchy problems with right-hand side function of the form $At^\alpha (\ln(t))^n$, $n \in \mathbb{Z}^+$.

$f(e^z)$ in the transformed equation is then $Az^n e^{\alpha z}$.

Guess for the particular solution is of the form $y_p(z) = (C_0 + C_1 z + \ldots + C_n z^n) e^{\alpha z}$.

Substitute $z = \ln(t)$ to obtain $y_p(t)$. 

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Result – Right-Hand Side a Product of a Monomial and a Positive Integer Power of ln(t)

Theorem

For the second order Euler-Cauchy problem,

\[ t^2 y'' + aty' + by = At^\alpha (\ln(t))^n, \quad t > 0, \]

where \( \alpha \in \mathbb{R} \) and \( n \in \mathbb{Z}^+ \), a particular solution is of the form

\[ y_p(t) = (C_0 + C_1 \ln(t) + \ldots + C_n(\ln(t))^n) t^\alpha. \]
Example: Find a general solution of 
\[ t^2 y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, \quad t > 0. \]

- Complementary solution: \( y_c = c_1 t + c_2 t^4 \).
- Particular solution: Solve \( t^2 y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t \).
  - Form: \( y_p = y_{p1} + y_{p2} \), where 
    \[ y_{p1} = (A + B(\ln(t)) + C(\ln(t))^2) \ t^2 \] (by Theorem 2), 
    \[ y_{p2} = Dt \ln(t) \] (by Theorem 1).
  - Plug \( y_p \) into the DE, collect terms, and equate coefficients to get 
    \( y_p = (-3 + 2 \ln(t) - 2(\ln(t))^2) \ t^2 + \frac{1}{3} t \ln(t) \).
- General solution: \( y = y_c + y_p \), so 
  \[
y(t) = c_1 t + c_2 t^4 + (-3 + 2 \ln(t) - 2(\ln(t))^2) \ t^2 + \frac{1}{3} t \ln(t) .
  \]
Example: Find a general solution of
\[ t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, \quad t > 0. \]

- Complementary solution: \( y_c = c_1 t + c_2 t^4 \).
- Particular solution: Solve \( t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t \).
  - Form: \( y_p = y_{p1} + y_{p2} \), where
    \[
    y_{p1} = (A + B(\ln(t)) + C(\ln(t))^2) t^2 \quad \text{(by Theorem 2)},
    \]
    \[
    y_{p2} = Dt \ln(t) \quad \text{(by Theorem 1)}.\n    \]
  - Plug \( y_p \) into the DE, collect terms, and equate coefficients to get
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    y_p = \left( -3 + 2 \ln(t) - 2(\ln(t))^2 \right) t^2 + \frac{1}{3} t \ln(t).
    \]
- General solution: \( y = y_c + y_p \), so
  \[
  y(t) = c_1 t + c_2 t^4 + \left( -3 + 2 \ln(t) - 2(\ln(t))^2 \right) t^2 + \frac{1}{3} t \ln(t).\n  \]
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\[ t^2 y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, \quad t > 0. \]

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    y_{p2} = D t \ln(t) \quad \text{(by Theorem 1)}.
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  - Plug \( y_p \) into the DE, collect terms, and equate coefficients to get
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- General solution: \( y = y_c + y_p \), so
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    \( y_p = (-3 + 2 \ln(t) - 2(\ln(t))^2) \ t^2 + \frac{1}{3} t \ln(t). \)
- General solution: \( y = y_c + y_p, \) so
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- General solution: \( y = y_c + y_p \), so
  \[
y(t) = c_1 t + c_2 t^4 + \left(-3 + 2 \ln(t) - 2(\ln(t))^2\right) t^2 + \frac{1}{3} t \ln(t).
  \]
Easily verified that the above approach also leads to a method of undetermined coefficients for Euler-Cauchy equations with the following right-hand side functions:

1. \( A \cos(k \ln t) \) or \( A \sin(k \ln t) \),
2. \( A t^\alpha \cos(k \ln t) \) or \( A t^\alpha \sin(k \ln t) \), and
3. \( A t^\alpha (\ln(t))^n \cos(k \ln t) \) or \( A t^\alpha (\ln(t))^n \sin(k \ln t) \).
Conclusions

- Straightforward to generalize this approach to higher order Euler-Cauchy equations.
- This “new” approach makes a good addition to the discussion of Euler-Cauchy problems in a differential equations course.