A Brief Introduction to Wavelets

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MATH 191T, Spring 2019
1. Motivation
2. Fourier Series and Fourier Transform
3. Windowed Fourier Transform
4. The Wavelet Transform
   - Continuous Wavelet Transform
   - Discrete Wavelet Transform and MRA
Wavelets have a number of applications across diverse fields, including:
- quantum physics,
- geology,
- electrical engineering, and
- astronomy.

They are needed to solve problems that cannot be handled with “classical” methods.
In the Beginning

- Define the set of functions on $\mathbb{R}$ that satisfy

\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty
\]

as $L^2(\mathbb{R})$.

- So, $L^2(\mathbb{R})$ is the set of square integrable functions.

- Consider a function $f \in L^2(\mathbb{R})$.

- Goal: write this (possibly complicated) function as a sum of well-known functions.
The Purpose of Writing Functions as Sums

- Suppose we want to analyze and synthesize a signal.
- Consider a musical score, such as the one below:
  \textbf{Goldberg Variations}
  
  \textit{BWV 988}
  
  Aria
  
  Johann Sebastian Bach

- One method of analysis is to separate the music into pure harmonic waves – this is the Fourier series.
Let $f$ be a function whose domain is $\mathbb{R}$ and whose range is the complex plane that

- has period $T$, i.e.,

$$f(t + T) = f(t),$$

and

- is square integrable, and so

$$\int_{t_0}^{t_0 + T} |f(t)|^2 dt < \infty.$$
The Fourier Series

- The Fourier series of $f$ can be defined as an infinite sum of sine and cosine terms, or it can be defined as

\[
\sum_{n=-\infty}^{\infty} c_n e^{2\pi i \omega_n t},
\]

where $\omega_n = \frac{n}{T}$.

- The coefficients are given by

\[
c_n = \frac{1}{T} \int_{t_0}^{t_0+T} e^{-2\pi i \omega_n t} f(t) \, dt.
\]
The Fourier Transform

- The Fourier Transform can be thought of as the continuous analogue of the Fourier series.
- To obtain the Fourier transform from the Fourier series:
  - Let $t_0 = \frac{T}{2}$ and take the limit as $T \to \infty$.
  - Denote by $\hat{f}$ the Fourier transform of $f$.
  - Then $\hat{f}$ is given by
    \[ \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) dt. \]

- The inverse Fourier transform is then defined by
  \[ f(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \hat{f}(\omega) d\omega. \]
Problems with Fourier Transform

- The Fourier transform and its inverse reproduce \( f(t) \) by a superposition of complex exponentials, \( e^{2\pi \omega t} \).
- The Fourier transform gives the coefficient function needed so that the (continuous) superposition, \( \int e^{2\pi \omega t} \hat{f}(\omega) d\omega \) equals \( f \).
- Observation: \( e^{2\pi \omega t} = \cos(2\pi \omega t) + i \sin(2\pi \omega t) \) is spread over all time.
- So, for a high-fidelity signal, infinitely many frequencies are needed (so we get destructive interference in the right places).
Problem: Reproducing a “sharp" function

Example: the Dirac delta “function"

Then \( \delta(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \hat{\delta}(\omega) \, d\omega = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \, d\omega. \)

Idea: the narrower the “spike" in a function, the more high-frequency exponentials needed to get the desired sharpness, but then we need more exponentials to cancel the effects of these functions elsewhere.
Improving the Representation of a Signal

- Instead of transforming the entire score of music into pure harmonics, we could try to change short segments of the score separately.

- Then each part of the signal is separated into a superposition of known functions of different frequency.

- But, each section of the signal has a different set of “amplitudes" associated with these functions.
Let \( g(u) \) be a square integrable function that is nonzero on the interval \([-T, 0]\) and zero for all other \( t \in \mathbb{R} \).

For each \( t \in \mathbb{R} \), define

\[
  f_t(u) = \overline{g}(u - t)f(u).
\]

Then \( f_t \) will be nonzero only on the interval \([t - T, t]\).

Define the windowed Fourier transform (WFT) of \( f \) as

\[
  \tilde{f}(\omega, t) = \hat{f}_t(\omega) = \int_{-\infty}^{\infty} e^{-2\pi \omega u} f_t(u) du
  = \int_{-\infty}^{\infty} e^{-2\pi \omega u} \overline{g}(u - t)f(u) du
\]
Some Properties of WFT

- The WFT is symmetric with respect to the time and frequency domains.
- Inverse WFT is defined by:

\[ f(u) = C^{-1} \int \int g_{\omega,t}(u)\tilde{f}(\omega,t) \, d\omega \, dt, \]

where \( C = \int |g(u)|^2 \, du. \)
What WFT Does

- WFT observes the signal $f(t)$ over the length of the window, $\bar{g}(u - t)$ such that the time parameter $t$ in $\tilde{f}(\omega, t)$ is no longer sharp – it represents a time window.
- In WFT, the frequency in $\tilde{f}(\omega, t)$ is not sharp either – it represents a frequency band associated with the band of the frequency window $\hat{g}(\nu - \omega)$ obtained from the Fourier transform of $\bar{g}(u - t)$.
- The product of the time and frequency windows is constant.
Example of Windowed Fourier Transform

- Ex. Gaussian window:

\[ g(t) = (2a) \frac{1}{4} e^{-\pi at^2}. \]

- Let \( a = \frac{1}{2\pi} \).

- Then, \( g \) looks like
Problems with Windowed Fourier Transform

- For the inverse WFT, $\tilde{f}(\omega, t)$ is the coefficient function used in the superposition of the functions $e^{2\pi \omega u} g(u - t)$.
- Since the window function $g$ has width $T$, $e^{2\pi \omega u} g(u - t)$ does, too.
- So, a spike in a function, with width much smaller than $T$ and located near $u = u_0$, can only be reproduced by superimposing $e^{2\pi \omega u} g(u - t)$ with $t \approx u_0$ and very many frequencies, $\omega$.
- If a function is very smooth, so its width is much larger than $T$, then many low-frequency functions $e^{2\pi \omega u} g(u - t)$ must be used over a wider time interval.
Problems with Windowed Fourier Transform (cont.)

- We see that the WFT introduces a scale into the analysis ($T$).
- This means that, in terms of a musical score, there will be sudden breaks between the segments of the score.
- In image processing, this could appear as an artificial edge.
Consider again a score of music.

Observe that there is localization in both the time and frequency domains.

- For tones with a high frequency, less time is required to produce them.
- For tones with a low frequency, more time is required to produce them.
The continuous wavelet transform is defined by

\[(T^{\text{wav}}f)(a, b) = |a|^{-\frac{1}{2}} \int f(t) \psi \left( \frac{t - b}{a} \right) dt.\]

- Assumption: \(\psi\) satisfies \(\int \psi(t) dt = 0\).
- The functions \(\psi^{a,b}(s) = |a|^{-\frac{1}{2}} \psi \left( \frac{s - b}{a} \right)\) are called wavelets.
- And \(\psi\) is called the mother wavelet.
In the definition of the wavelet transform, as the parameter $a$ changes, the $\psi^{a,0}(s) = |a|^{-\frac{1}{2}} \psi(\frac{s}{a})$ cover different frequency ranges.
- Large values of $a$ correspond to low frequencies.
- Small values of $a$ correspond to high frequencies.

Changing the parameter $b$ changes the time-localization center (since $\psi^{a,b}(s)$ is localized around $s = b$).
By the definition of the wavelet functions, they have time-widths (or space-widths) related to their frequency:

- high frequency = narrow function,
- low frequency = broad function.

So, wavelets give us a mathematical way to represent a signal (e.g., a radio signal) by:

- decomposing according to its frequency components, and
- analyzing each component with a resolution matching its scale.

This means wavelets can “zoom in” on short frequency bursts and “zoom out” on longer frequency signals.
Types of Wavelet Transforms

- The continuous wavelet transform
- The discrete wavelet transform
  - Discrete systems with redundancy (frames)
  - Orthonormal (or other) bases of wavelets
- We will briefly touch upon the continuous wavelet transform and orthonormal bases of wavelets (multiresolution analysis).
Continuous Wavelet Transform

- Here, $\psi^{a,b}(x) = |a|^{-\frac{1}{2}} \psi\left(\frac{x-b}{a}\right)$ and

$$\left(T^{\text{wav}}\right)(a, b) = \int f(t) \overline{\psi^{a,b}}.$$  

- A function can be reconstructed from its wavelet transform by means of the following formula:

$$f = C_\psi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} \left(\left(T^{\text{wav}}\right)(a, b)\right) \psi^{a,b} db da,$$  \quad (1)

where $C_\psi^{-1} = 2\pi \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 |\xi|^{-1} d\xi$.

- For (1) to make sense, we need $C_\psi^{-1} < \infty$. 

An Example of a Wavelet: $\psi(t) = (1 - t^2)e^{\frac{-t^2}{2}}$

(Also known as the Mexican hat function)

Note: This wavelet is often used in analyzing vision. It is scaled (changing the maximum and minimum values) and dilated (changing the width) to analyze a signal.
Another Example: Morlet Wavelet

\[ \psi(t) = \pi^{-\frac{1}{4}} \left( e^{-i\omega_0 t} - e^{-\frac{i\omega_0^2 t}{2}} \right) e^{-\frac{t^2}{2}} \], with

\[ \omega_0 = \pi \sqrt{\frac{2}{\ln 2}} \]

Real part of \( \psi(t) \)

Imaginary part of \( \psi(t) \)
Example Application – the Radio

- When we listen to music on the radio, what do we hear?
- First, the sounds must be compressed, sent over the airwaves, then uncompressed.
- In uncompression, external noise must be removed from the signal, so we hear only the music.
- Amplifiers are responsible for this noise removal.
- Wavelets give a way to compress a signal and then uncompress it without keeping high-frequency noise, and without losing any details of the original signal.
Why Do We Need Discrete Wavelets?

- Now, looking at a digital recording (or image), information is not in continuous form.

- Discrete wavelets allow analysis of sequences of information in the same way that continuous wavelets analyze continuous functions.

- Of course, discrete wavelets can be used to analyze continuous functions also, by sampling the function at appropriate points, then transforming it.
Multiresolution Analysis (MRA): Idea

- Typically, there are several levels of decomposition of a signal.
- The signal is broken up into “approximation” parts and “detail” parts, which are then used to reconstruct the signal.
Multiresolution Analysis (MRA): Some Mathematical Detail

- Consists of sequence of closed subspaces \( V_j \) of \( L^2(\mathbb{R}) \)

\[
\cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset L^2(\mathbb{R})
\]

satisfying certain properties, including:

- For every integer \( j \),
  \[
  V_{j+1} \subset V_j.
  \]

- A function \( f \) is in one of the sets \( V_j \) if and only if, for every integer \( j \),
  \[
  f(2^j \cdot) \in V_0.
  \]

- The only function contained in all \( V_j \)’s is \( f(x) = 0 \).
If \( f \in V_0 \), then \( f(x + n) \in V_0 \) for every integer \( n \).

There exist functions \( \phi \in V_0 \), called scaling functions, such that

\[
\{ \phi(x - n) \mid n \in \mathbb{Z} \}
\]

forms an orthonormal basis of \( V_0 \).

This means that \( \{ \phi_{j,n} = 2^{-j/2} \phi(2^{-j}x - n) \mid n \in \mathbb{Z} \} \) forms an orthonormal basis for \( V_j \), for all \( j \), so any function in \( V_j \) can be written as a weighted sum of the \( \phi_{j,n} \).
For each integer $j$, define the set $W_j$ so that every function in $W_j$ is orthogonal to every function in $V_j$, and every function in $V_{j-1}$ can be written as a sum of two functions, one in $V_j$ and one in $W_j$.

We can find $\psi \in W_0$, (mother wavelet) such that

$$\{\psi(\cdot - k) \mid k \in \mathbb{Z}\}$$

is an orthonormal basis for $W_0$.

Then $\{\psi_{j,n}\}$ form an orthonormal basis of $W_j$, for all $j$. 

Multiresolution Analysis (cont.)
Example of Multiresolution Analysis: the Haar MRA

\[ \phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise} \end{cases} \quad \psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leq x < 1, \\ 0 & \text{otherwise} \end{cases} \]

(So, \( V_j \) for Haar MRA is the space of piecewise constant functions.)
One Way to Use an MRA

- We may represent the discrete wavelet and scaling operators as matrices.
- Store the data from a signal (or function) in a vector, and then multiply by the appropriate matrix to get the “approximation” parts and the “detail” parts.
- In two dimensions, the operators are constructed in a certain way as products of the one dimensional operators; the principle properties of the operators do not change.
Example: Compression of an Image

- The idea behind image compression is to take an image of a given number of pixels and store it using the smallest amount of memory possible, e.g., in a smart phone.

- A simple way to compress an image is to decompose the image using wavelets, then use thresholding to choose which values to delete.
Thresholding

- Choose the smallest components from the wavelet decomposition based on a given tolerance and remove them.
- The threshold may also be defined based on the percent compression desired.
- Due to the properties of wavelets, significant compression can be realized with little effect on the reconstructed image.
Reconstruction of an Image

- The procedure used to decompose the image is in some sense “reversed” based on established mathematical formulas.

- We’ll look at examples of images compressed and reconstructed using different wavelet families, including Haar wavelets and wavelets constructed by Ingrid Daubechies.
Some References


