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Uniquely Decodable Ternary Codes via Augmented Sylvester-Hadamard Matrices

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***Abstract*—In this paper, we consider the problem of designing uniquely decodable (UD) ternary code sets for low-density spreading code-division multiple-access (LDS-CDMA) systems. The proposed code set provides unique decodability for a number of users K that is greater than the length of the code but lower than the theoretical maximum, i.e., $K < K^t_{\max}$*. Simulation results show that with channel encoded scenarios (e.g., turbo, LDPC, polar, etc.) the proposed UD code sets achieve 4 dB or greater performance improvement at a bit error rate (BER) of 10^{-3} compared to other competing code sets in an additive white Gaussian noise (AWGN) and flat Rayleigh fading channels.**

***Index Terms*—Uniquely decodable codes, ternary overloaded uniquely decodable codes, overloaded CDMA, non-orthogonal multiple-access (NOMA), low-density spreading signatures (LDS), sparse-code multiple-access (SCMA).**

I. Introduction

The design of uniquely decodable code sets for overloaded synchronous code-division multiple-access (CDMA) where the number of multiplexed signals K is greater than the spreading (or signature) code length L has been studied in [1]–[15]. An overloaded CDMA system using code set $\mathbf{C}$ is considered to be uniquely decodable (UD) if, in a noiseless multiplexed transmission, $\mathbf{C}\mathbf{x}_1 \neq \mathbf{C}\mathbf{x}_2$ for all possible $K \times 1$ vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, where $\mathbf{x}_1 \neq \mathbf{x}_2 \in \{\pm 1\}^{K\times 1}$. In other words, a UD matrix is injective in nature or there exists a one-to-one mapping between the input and output.

Uniquely decodable code set construction for overloaded CDMA for the noiseless channel, where $\mathsf{F}_3(L)$ represents the maximal number of L-dimensional vectors for a given length L that matrix can have and still be uniquely decodable is related to the coin-weighing problem, one of the problems that is discussed by Erdős and Rényi in [3]. The explicit construction techniques were proposed for these UD code sets of binary $\{0, 1\}$, antipodal $\{\pm 1\}$, and ternary $\{0, \pm 1\}$ chips in [1], [4], [5], [5]–[15], respectively, most of which are recursive in nature. To the best of our knowledge, it is worth mentioning that the maximum number of vectors of the explicit constructions of binary, antipodal and ternary code sets are $K^b_{\max} = \gamma(L+1)^\dagger$, $K^a_{\max} = \gamma(L)+1$ and $K^t_{\max}{}^\ddagger = (i+2)2^{(i-1)}$, respectively.

The code sets, which are primarily designed for the noiseless channel, can be decoded by relatively fast recursive deterministic decoders that require very little computation. In noisy channels, one may apply the optimal decoder, such as, maximum likelihood (ML); however, the computational complexity grows with the code length and it is not very practical. Overloaded matrices over the ternary alphabet are introduced by Mashayekhi and Marvasti in [14] with a fast logical decoder, requiring only a few comparisons to decode the received vectors. Similarly, Singh *et al.* in [15] propose overloaded code sets over the ternary alphabet that have twin tree structured cross-correlation hierarchy with a simple multi-stage detection.

Multiuser M-ary $\{0, 1, 2, \ldots, M\}$ UD code sets were investigated by Lu *et al.* in [16], [17] for the noiseless channel. Other works have applied UD codes in various communication systems. As an example, Yu *et al.* in [18], [19], Kulhandjian *et al.* in [20] and Buch *et al.* in [21] proposed UD codes as a solution for physical-layer network coding (PNC), specifically, in wireless cooperation communication systems.

All of the multiple access concepts for overloaded systems were introduced in order to serve a number of excess users beyond the available resources. These multiple access schemes are characterized by non-orthogonal multiple access (NOMA) techniques [22] that have received significant attention for the fifth generation (5G) cellular networks [23]. Recently, several NOMA solutions have been actively investigated [24], which can be divided into two main categories, namely power-domain and code-domain NOMA. A few of the strong contenders of code-domain NOMA are low-density spreading aided CDMA (LDS-CDMA) [25] and sparse code multiple access (SCMA) [23], [26]. LDS-CDMA [25] must generally guarantee to be UD code set [27], which means code sets have non-zero Euclidean distance. The design of LDS type matrices offers flexible resource allocation,

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*$K^t_{\max}$ is the maximum number of users for which there are existing code sets that achieves UD for a given code length and there is no UD code set construction if the number of users are greater.

†where $\gamma(n)$ function is the number of ones in the binary expansion of all positive integers less than n. As an example, $\gamma(8) = 12$.

‡Superscripts ^{t}, ^{a}, ^{b} refer to ternary, antipodal and binary code sets, respectively.

performs better in terms of handling the multiple access interference (MAI) that exists in rank-deficient systems and has low-complexity receivers compared to conventional CDMA. LDS-CDMA, may also be appropriate for IoT communications [22] and it is also considered as a potential candidate for uplink machine-type-communications (mMTC) [22].

Inspired by these attractive features of UD code sets, this paper proposes the recursive construction of UD ternary code sets as a potential candidate for LDS-CDMA NOMA scheme. Simulation results in terms of bit error rate (BER) demonstrate that the performance of the proposed UD code set with channel encoding outperforms the existing ternary codes. In the case of dispersive fading channels, we can potentially equalize the channel effects by eliminating them using channel precoding as described in [28].

The rest of the paper is organized as follows. In Section II, we present the construction of the uniquely decodable ternary code sets followed by the minimum distance in Section III and multiuser detector in Section IV, respectively. In Section V, we present our simulation methodology and results before presenting our conclusion in Section VI.

The following notations are used in this paper. All boldface lower case letters indicate column vectors and upper case letters indicate matrices, $()^T$ denotes transpose operation, $\mathbb{C}$ denotes the set of all complex numbers, and $|\cdot|$ denotes complex amplitude, respectively.

II. Recursive Code Construction

In this manuscript, we propose the design of ternary matrices based on augmented Sylvester-Hadamard matrices. More explicitly, in the matrices $\mathbf{C}_{L\times K}$ that can be written as $\mathbf{C}_{L\times K} = [\mathbf{H}_L \mathbf{V}_L]$, where $\mathbf{V}_L \in \{0, \pm 1\}^{L\times(K-L)}$ and $\mathbf{H}_L$ is a Sylvester-Hadamard matrix of order $L = 2^p$. We recall that the Sylvester-Hadamard matrix of order 2 is $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{H}_{2^{p+1}} = \begin{bmatrix} \mathbf{H}_{2^p} & \mathbf{H}_{2^p} \\ \mathbf{H}_{2^p} & -\mathbf{H}_{2^p} \end{bmatrix}$, $p = 1, 2, ...$. Then, for any $p = 1, 2, ...$, $\mathbf{H}_{2^p}\mathbf{H}_{2^p} = 2^p \mathbf{I}_{2^p\times 2^p}$, where $\mathbf{I}_{N\times N}$ is the $N\times N$ identity matrix. For the case when $L = 4$, we can augment the $\mathbf{H}_4$ matrix to form an UD ternary matrix, $\mathbf{C}_{4\times 5}$ as follows,

$$\mathbf{C}_{4\times 5} = \begin{bmatrix} + & + & + & + & - \\ + & - & + & - & 0 \\ + & + & - & - & 0 \\ + & - & - & + & 0 \end{bmatrix}. \tag{1}$$

It is not hard to find all possible vectors[§] that can be appended to $\mathbf{H}_4$ and still satisfy the UD property. Instead, our goal is to design UD code matrices in a recursive fashion. Let us consider UD code matrix for the case $L = 8$. We need to select a ternary UD matrix, $\mathbf{V}_8$, that when appended to $\mathbf{H}_8$ it will still satisfy the UD property. Moreover, $\mathbf{V}_8$ should also serve as the seed for generating $\mathbf{V}_L$ matrices for any L in recursive manner. In order to satisfy those requirement one possible option is to choose the following matrix,

$$\mathbf{V}_8 = \begin{bmatrix} -\alpha_2 & \alpha_2 & \alpha_6 & -\alpha_5 & \alpha_5 \\ \alpha_8 & \alpha_8 & \alpha_8 & -\alpha_5 & -\alpha_5 \end{bmatrix}, \tag{2}$$

where $\alpha_1 = [1,0,1,0]^T$, $\alpha_2 = [1,0,-1,0]^T$, $\alpha_3 = [0,1,0,-1]^T$, $\alpha_4 = [0,0,0,0]^T$, $\alpha_5 = [1,0,0,0]^T$, $\alpha_6 = [0,-1,0,-1]^T$, $\alpha_7 = [1,1,0,0]^T$, and $\alpha_8 = [0,0,1,1]^T$. Therefore, the resulting matrix is $\mathbf{C}_8 = [\mathbf{H}_8 \mathbf{V}_8]$. We are now ready to propose a general $L_i \times K_i$ code set design for $L_i = 2^i$ and $K_i = i2^{i-1} + 1$, where $i \geq 4$. Given the $\mathbf{V}_8$ matrix in (2), we can recursively construct matrix $\mathbf{V}_{L_i}$ as follows

$$\mathbf{V}_{L_i} = \begin{bmatrix} \mathbf{V}_{L_i/2} & \mathbf{V}_{L_i/2} & \mathbf{R}_i \\ \mathbf{V}_{L_i/2} & -\mathbf{V}_{L_i/2} & \mathbf{0}_{L_i/2} \end{bmatrix}, \tag{3}$$

where $\mathbf{V}_{L_i/2}$ is a $2^{i-1} \times (i-3)2^{i-2} + 1$ matrix constructed in the previous step. The matrix $\mathbf{R}_i = [\mathbf{r}_0, ..., \mathbf{r}_{M-1}]^T$ is formed as follows, for $j = 0$,

$$\begin{aligned} \mathbf{r}_j &= \left[-\alpha_2, \alpha_2, \alpha_6, \alpha_4, \alpha_7, \alpha_7, \alpha_7, \Lambda_4^{t_0}, \right]^T \\ \mathbf{r}_{j+1} &= \left[\alpha_8, \alpha_8, \alpha_8, \alpha_3, -\alpha_2, \alpha_2, \alpha_6, \Lambda_4^{t_0}\right]^T \end{aligned}$$

and for $j = 2, 4, 6, \ldots 2^{i-3} - 2$,

$$\begin{aligned} \mathbf{r}_j &= \left[\Lambda_4^{t_1}, \alpha_4, -\alpha_2, \alpha_2, \alpha_6, \alpha_4, \alpha_7, \alpha_7, \alpha_7, \Lambda_4^{t_2}\right]^T \\ \mathbf{r}_{j+1} &= \left[\Lambda_4^{t_1}, \alpha_3, \alpha_8, \alpha_8, \alpha_8, \alpha_3, -\alpha_2, \alpha_2, \alpha_6, \Lambda_4^{t_2}\right]^T, \end{aligned}$$

where $t_0 = 2^3(2^{(i-4)} - 1)$, $t_1 = 4j - 1$, $t_2 = 2^{(i-4)} - j/2 - 1$, $\Lambda_4^t = \alpha_4 \otimes \mathbf{1}_{4,t}$, $\otimes$ denotes the Kronecker product and $\mathbf{1}_{m_0,m_1}$ denotes all-one matrix of size $(m_0 \times m_1)$.

$$\mathbf{C}_{8\times 13} = \begin{bmatrix} + & + & + & + & + & + & + & + & - & + & 0 & - & + \\ + & - & + & - & + & - & + & - & 0 & 0 & - & 0 & 0 \\ + & + & - & - & + & + & - & - & + & - & 0 & 0 & 0 \\ + & - & - & + & + & - & - & + & 0 & 0 & - & 0 & 0 \\ + & + & + & + & - & - & - & - & 0 & 0 & 0 & - & - \\ + & - & + & - & - & + & - & + & 0 & 0 & 0 & 0 & 0 \\ + & + & - & - & - & - & + & + & + & + & + & 0 & 0 \\ + & - & - & + & - & + & + & - & + & + & + & 0 & 0 \end{bmatrix}$$

Fig. 1. Errorless signature set $\mathbf{C}$ for an overloaded system with $L = 8$ and $K = 13$.

III. Minimum Distance of Code sets

The Manhattan Distance [29] between two L-dimensional vectors $\mathbf{y}_i$ and $\mathbf{y}_j$ for $i \neq j$ is equivalent to their (ℓ_1)-norm, which is defined as

$$d_L(\mathbf{y}_i, \mathbf{y}_j) = \sum_{t=1}^{L} |y_{i,t} - y_{j,t}|. \tag{4}$$

Then the general minimum Manhattan distance of received vectors for a given ternary code set can be formulated by

$$d_{min}(\mathbf{C}) = \underset{\substack{\mathbf{x}_i, \mathbf{x}_j \in \{\pm 1\}^{K\times 1} \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i, \mathbf{y}_j = \mathbf{C}\mathbf{x}_j}}{\text{argmin}} d_L(\mathbf{y}_i, \mathbf{y}_j). \tag{5}$$

[§]It can be achieved by exhaustive search of $3^4/2 - 1$ vectors excluding zero vector and complement (e.g., multiplication by minus one) vectors.

$$
\mathbf{C}_{16\times 33}=\begin{bmatrix}
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+&-&-&+&-&+&+&-&-&+&+&-&+&-&-&+&+&+&+&0&0&-&-&-&0&0&0&0&0&0&0&0&0
\end{bmatrix}
$$

Fig. 2. Errorless signature set $\mathbf{C}$ for an overloaded system with $L = 16$ and $K = 33$ an overload factor of 206.25%.

Theorem 1. *Let $\mathfrak{C} \in \{0, \pm 1\}^{L\times K}$ represent the set of all ternary matrices constructed by distinct* [e] *columns, then the minimum distance of the code set, $\delta(\mathfrak{C})$, is equal to 2, where*

$$\delta(\mathfrak{C}) = \underset{\mathbf{C}'\in\mathfrak{C}}{\operatorname{argmin}}\, d_{min}(\mathbf{C}'). \tag{6}$$

Proof. Let us assume that $\delta(\mathfrak{C}) = d_{\min}(\mathbf{C}) = d_L(\mathbf{y}_n, \mathbf{y}_m)$, where $\mathbf{y}_n = \mathbf{C}\mathbf{x}_n$, $\mathbf{y}_m = \mathbf{C}\mathbf{x}_m$, $\mathbf{y}_n, \mathbf{y}_m \in \mathcal{N}^{L\times 1}$, $\mathcal{N} \in \{\pm K, \pm(K-1), ...\}$, and $\mathbf{x}_n, \mathbf{x}_m \in \{\pm 1\}^{K\times 1}$. The minimum value obtained when the difference vector $\mathbf{y} = \mathbf{y}_n - \mathbf{y}_m = \mathbf{C}(\mathbf{x}_n - \mathbf{x}_m) = \mathbf{C}\bar{\mathbf{x}}$ has only one non-zero element $y_c \neq 0$, $y_{n,c} \neq y_{m,c}$ having the lowest magnitudes $y_{n,c}, y_{m,c} \in \{\pm 1\}$[f], and $L-1$ zeros $y_i = 0$, $y_{n,i} = y_{m,i}$ for $i \neq c$, where $\bar{\mathbf{x}} \in \{0, \pm 2\}^{K\times 1}$. In order to achieve only one non-zero element of $\mathbf{y}$ then $\mathbf{c}^i\bar{\mathbf{x}} = 0$ for $\forall i \in \{1, \ldots, L\}$ except $i \neq c$, where $\mathbf{c}^i$ is the i-th row of matrix $\mathbf{C}$. One possible option is to have all zeros but one non-zero element in $\bar{\mathbf{x}}$. In this case, the matrix $\mathbf{C}$ must have a column $\mathbf{c}_j$ with $c_{j,i} = 0$ for $\forall i \in \{1, \ldots, L\}$ except $i \neq c$, where $1 \leq j \leq K$ to result in a vector $\mathbf{y}$ having only one non-zero element. In other words, $\mathbf{c}_j$ contains only one ± 1 element. Therefore, having the lowest magnitude of $y_{n,c} = 1$ and $y_{m,c} = -1$ or $y_{n,c} = -1$ and $y_{m,c} = 1$ results in $d_{\min}(\mathbf{C}) = |y_{n,c} - y_{m,c}| = 2$. □

Now that we proved that $\delta(\mathfrak{C}) = 2$, we will try to find $d_{\min}(\mathbf{C})$ of our proposed UD code sets $\mathbf{C} \in \mathcal{C} \subset \mathfrak{C}$, where $\mathcal{C} \in \{0, \pm 1\}^{L\times K}$ is the set of all the ternary UD code sets. Let us follow the option of having all zeros but one non-zero elements in $\bar{\mathbf{x}}$. In the case of $L = 4$, the column $\mathbf{c}_5$ has only the first element non-zero. If $\bar{\mathbf{x}} = [0, 0, 0, 0, -2]^T$ then the difference vector is $\mathbf{y} = [2, 0, 0, 0]^T$. Note that even if we substitute the $\mathbf{c}_5$ by $-\mathbf{c}_5$ we can still obtain the same $\mathbf{y}$ with $\bar{\mathbf{x}} = [0, 0, 0, 0, 2]^T$. Based on our construction in (2), we look at the case of $L = 8$. If $\bar{\mathbf{x}} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0]^T$ then the difference vector is $\mathbf{y} = [2, 0, 0, 0, 2, 0, 0, 0]^T$. Unlike the case of $L = 4$, the vector $\mathbf{y}$ has two non-zero elements, which results $d_{min}(\mathbf{C}) = 4$. Alternatively, observe that the columns 12-th and 13-th, $[-\alpha_5, -\alpha_5]^T$ and $[\alpha_5, -\alpha_5]^T$ of the $\mathbf{C}$, all the elements are equal except at the 1-st element in which they differ. If we select $x_{n,12} \neq x_{m,12}$, $x_{n,13} \neq x_{m,13}$, and $x_{n,i} = x_{m,i}$ for all $i \notin \{12, 13\}$ then $y_{n,1} = 2$ and $y_{m,1} = -2$ or $y_{n,1} = -2$ and $y_{m,1} = 2$ will result in $d_L(\mathbf{y}_n, \mathbf{y}_m) = 4$. With this specific observation together with the Theorem 1, we conclude that $d_{min}(\mathbf{C}) = 2$ and $d_{min}(\mathbf{C}) = 4$ for the cases of $L = 4$ and $L = 8$. From this observation, we learn that if a column has only two ± 1 elements or any two columns differ at one element in a UD code set, we assure that $d_{\min}(\mathbf{C}) = 4$. Similarly, for $L = 16$ columns 30-th, $[0, \alpha_3, 0, 0, 0]^T$ with the 6-th and 8-th elements having $+1$ and -1 values. Due to our recursive construction in (3) for $L = 2^p$, where $p = 5, 6, ...$ columns $(p-1)2^{(p-1)} + 6$ has two non-zero elements only. Therefore, all the UD code set generated by (3) has $d_{min}(\mathbf{C}) = 4$.

IV. Multiuser Detection

In the overloaded (i.e., $K > L$) synchronous code-division multiple-access application of interest, each user multiplexes its own symbol by multiplying it with the signature and then transmitting it through the channel after carrier modulation. In the case of additive white Gaussian noise (AWGN) channel, the received vector can be expressed by

$$\begin{aligned}\mathbf{y} &= \sum_{k=1}^{K} \mathbf{c}_k d_k x_k + \mathbf{n} \\ &= \mathbf{C}\mathbf{D}\mathbf{x} + \mathbf{n},\end{aligned} \tag{7}$$

where d_k is the k-th user's amplitude, $x_k \in \mathcal{X}_k$ is the k-th user's symbol to be transmitted from the constellation alphabet, $\mathcal{X}_k$, $\mathbf{c}_k \in \{\pm 1, 0\}^{K\times 1}$ is the spreading waveforms, $\mathbf{n} \in \mathbb{C}^{L\times 1}$ is an L-dimensional complex-valued AWGN vector with variance of σ^2 and $\mathbf{D}$ is a diagonal matrix with users' amplitude.

The objective of the receiver is the following; given the received vector $\mathbf{y}$ and $\mathbf{C}$ recover the user data $\hat{\mathbf{x}}$ such that the mean square error $E\{||\mathbf{x} - \hat{\mathbf{x}}||^2\}$ is minimized. It is known that obtaining the ML solution is generally NP-hard [30]. The suboptimal detectors are more preferable in terms of complexity algorithm compared to ML. Some of the suboptimal detectors include matched filter (MF), and zero-forcing (ZF), minimum mean square error (MMSE),

[e] Not only the columns are required to be distinct but we assume any column multiplication with minus one should result in distinct columns as well.

[f] In the case of antipodal matrices $\mathbf{C} \in \{\pm 1\}^{L\times K}$, the lowest magnitude is $y_{n,c}, y_{m,c} \in \{\pm 2\}$ and the number of ± 2 elements in $\bar{\mathbf{x}}$ must be even for the inner product, $\mathbf{c}^i\bar{\mathbf{x}}$, to be zero.

parallel interference cancellation (PIC) [31], probabilistic data association (PDA) [32], etc. In this study, we consider two low-complexity detectors, MMSE-PIC detector, which is based on the MMSE and PIC criteria, and PDA. In case of flat Rayleigh fading and multipath fading channels we utilize the low-complexity multiuser detection (MUD) detectors, which are proposed in [28].

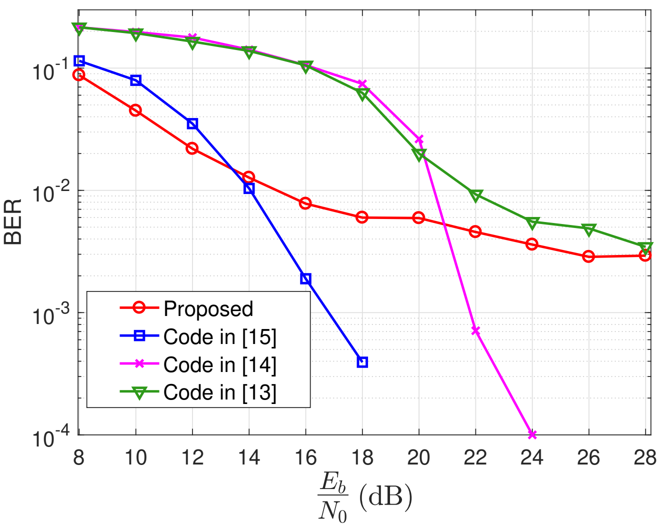

Fig. 3. Uncoded $\mathbf{C}_{4\times 5}$ using MMSE-PIC detector.

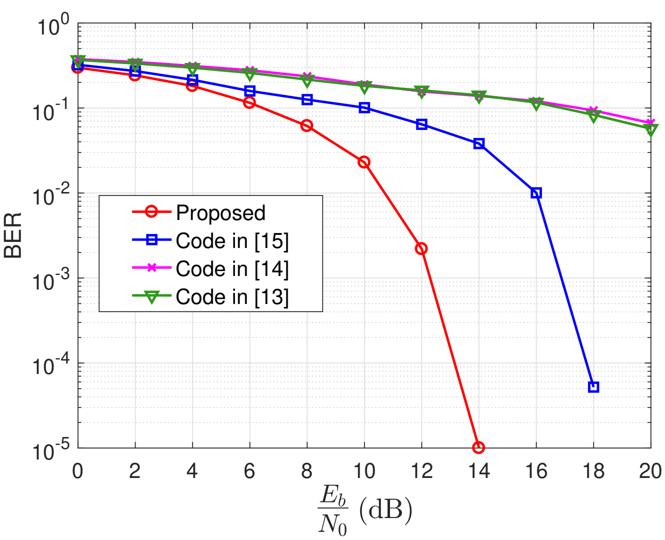

Fig. 4. LDPC $\mathbf{C}_{4\times 5}$ using PDA detector.

V. Simulation Results

In this section, we evaluate the performance of LDS-CDMA NOMA over AWGN and flat Rayleigh fading channels employing our proposed ternary uniquely decodable codes at the physical layer. All the simulations at the physical layer of the proposed scheme is performed in Matlab. We consider wireless transmission with the number of users $K = 5$ and $K = 13$. Each user k spreads its data $x_k \in \{\pm 1\}$ using binary-phase shift keying (BPSK) modulation. At the receiver, MUD is performed using MMSE-PIC [28] and PDA [32]. In addition to that in our simulations, we have included code constructions from [13]–[15]. In Fig. 3, we plot the BER performance over AWGN channel averaged over all the different users for the proposed UD code set $\mathbf{C}_{4\times 5}$ and we compare them with the $\mathbf{C}_{4\times 7}$, $\mathbf{C}_{4\times 6}$ and $\mathbf{C}_{4\times 8}$ constructions presented in [14], [15] and [13]. The BER performance of the proposed code set with the uncoded case outperforms other code sets up to signal to noise ratio (SNR) of 13 dB after which it degrades, as shown in Fig. 3. The reason of such performance is related to linear separability margin described in [33], where the proposed code set has the largest value compared to other code sets.

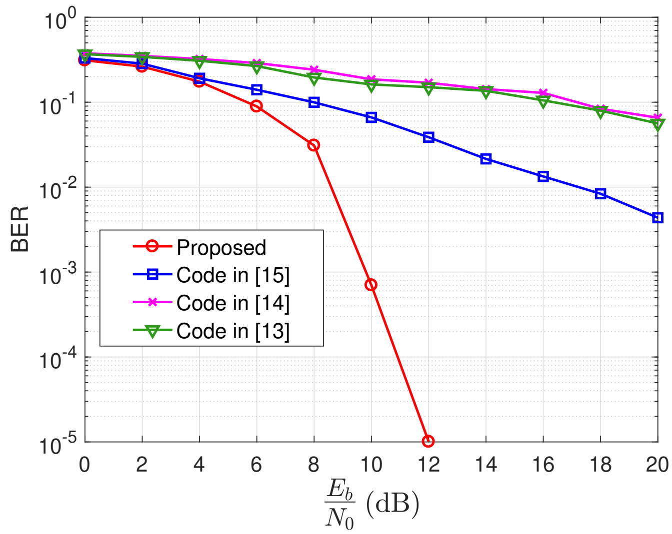

Fig. 5. Turbo $\mathbf{C}_{4\times 5}$ using PDA.

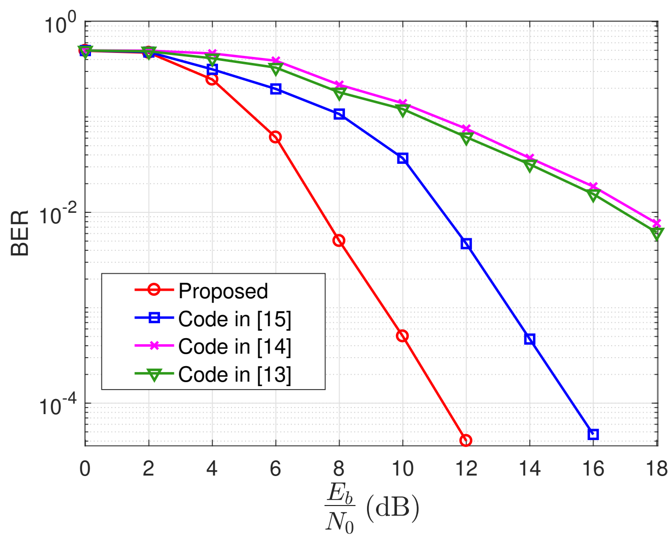

Fig. 6. Polar $\mathbf{C}_{4\times 5}$ using MMSE-PIC.

For channel encoded cases, we utilize a custom semi-random parity check matrix generator [34] for LDPC, long-term evolution (LTE) turbo code described in [35] and calculating the Bhattacharyya parameters of bit channels construction method [36] for polar codes. The construction of the LTE turbo interleaver is based on the quadratic permutation polynomial (QPP) scheme of [35]. For all three channel encoding cases we used a code rate of $1/3$ with input message block lengths of 320 bits and output code length of 972 for both the LDPC and LTE turbo code and 340 input, 1024 output bits for polar encoding. In Fig. 4, Fig. 5 and Fig. 6 we present simulation results over the flat Rayleigh, AWGN and flat Rayleigh channels for the LDPC, turbo and polar channel encoded cases. The proposed code set $\mathbf{C}_{4\times5}$ outperforms in terms of BER compared to other constructions. Similarly, in Fig. 7, we plot the BER performance over flat Rayleigh channel averaged over all the different users for the proposed $\mathbf{C}_{8\times13}$ and we compare them with the $\mathbf{C}_{8\times15}$, $\mathbf{C}_{8\times14}$ and $\mathbf{C}_{8\times17}$.

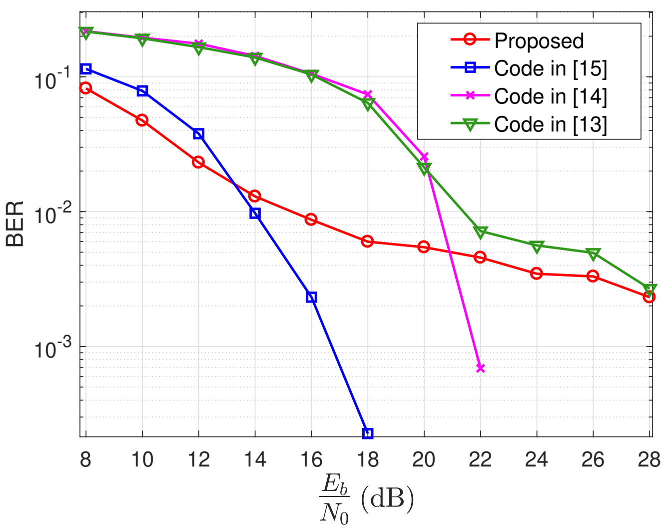

Fig. 7. Uncoded $\mathbf{C}_{8\times13}$ using PDA detector.

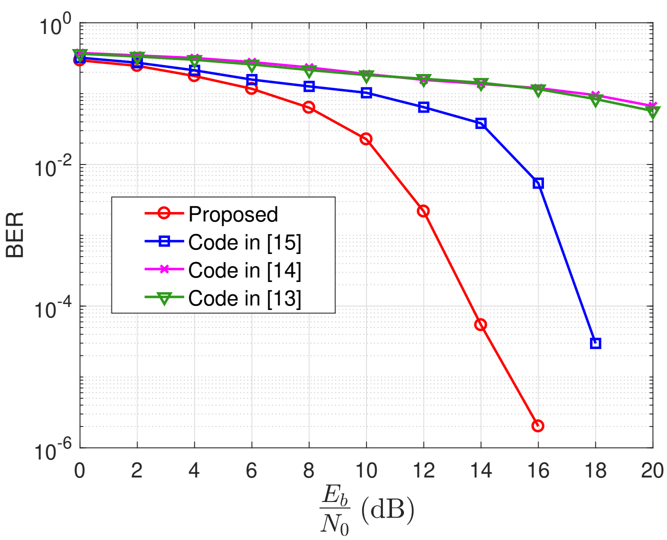

Fig. 8. LDPC $\mathbf{C}_{8\times13}$ using MMSE-PIC detector.

Again for the uncoded case the proposed $\mathbf{C}_{8\times13}$ performs better up to 13 dB SNR. To illustrate the gain in terms of BER rate, we present over the AWGN, flat Rayleigh, and AWGN channels for the LDPC, turbo and polar channel encoded cases in Fig. 8, Fig. 9 and Fig. 10, respectively.

There is a trade-off between the number of users, K, and BER performance, however, we can observe from Figs. 4, 5, 6, 8, 9 and 10 that our propose UD code set outperforms the code constructions in [14], [15] and [13] in terms of BER. It is obvious that overloaded UD code sets can potentially increase the user capacity by more than double for large L.

VI. Conclusion

In this paper, we have introduced new uniquely decodable (UD) ternary code sets for low-density spreading code-division multiple-access (LDS-CDMA) systems.

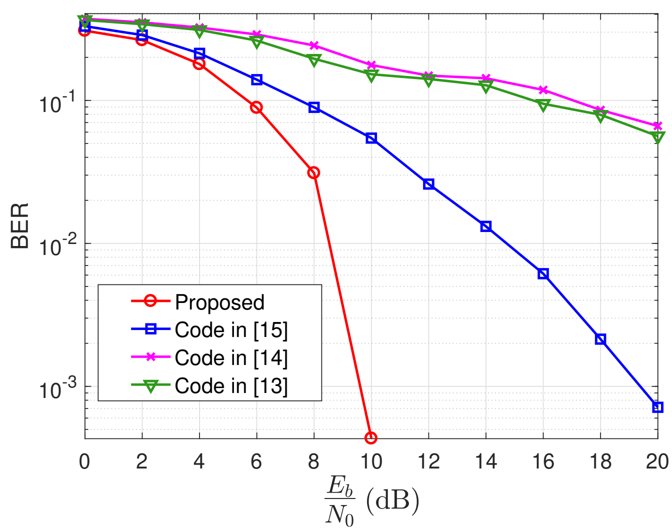

Fig. 9. Turbo $\mathbf{C}_{8\times13}$ using MMSE-PIC detector.

In comparison to the current state-of-the-art ternary code sets, which have low-complexity detectors, the proposed construction outperformed in terms of bit error rate (BER).

Simulation results show that with the channel encoded scenarios (e.g., turbo, LDPC, polar, etc.) the proposed UD code sets achieve 4 dB or greater performance improvement at a BER of 10^{-3} compared to other competing code sets in an additive white Gaussian noise (AWGN) and flat Rayleigh channels. In our simulations studies, we only included binary-phase shift keying (BPSK) signaling.

For our future work, we are investigating UD code sets for higher constellations such as quadrature phase-shift keying (QPSK), quadrature amplitude modulation (QAM), etc.

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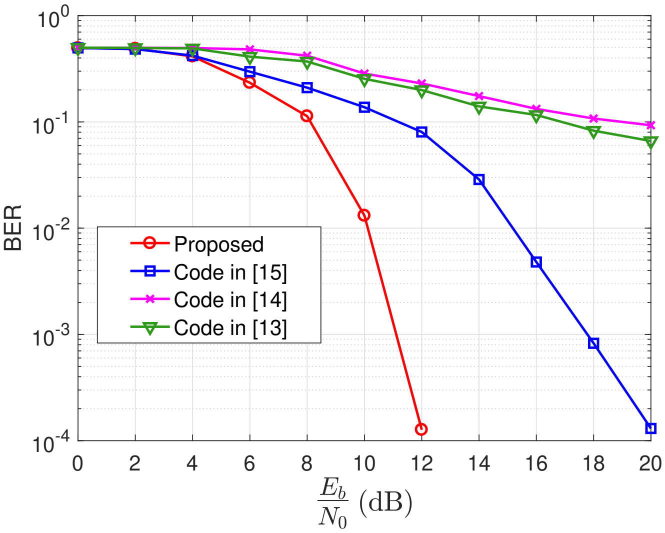

Fig. 10. Polar $\mathbf{C}_{8\times 13}$ using PDA detector.

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