Multidisciplinary Discussion on 5G from the Viewpoint of Algebraic Combinatorics

Michel Kulhandjian[†], Levon Aslanyan[‡], Hasmik Sahakyan[‡], Hovannes Kulhandjian[§] and Claude D'Amours[†]
[†]School of Electrical Engineering and Computer Science
University of Ottawa, Ottawa, Ontario, K1N 6N5, Canada
mkk6@buffalo.edu, cdamours@uottawa.ca
[‡]Institute for Informatics and Automation Problems
National Academy of Sciences, Yerevan, 0014, Armenia
{lasl, hsahakyan}@sci.am
[§]Department of Electrical and Computer Engineering
California State University, Fresno, Fresno, CA 93740, U.S.A.
hkulhandjian@csufresno.edu

***Abstract*—In this paper, we present some of the interesting questions raised in the "5G and Beyond" workshop discussions. We propose to explore alternative solutions in other discipline such as algebraic combinatorics and discrete optimization. Construction of matrices that possess certain desirable properties in discrete tomography can be exploited for the 5th generation (5G) multiuser waveform design. Moreover, we propose to investigate thoroughly the random graph theory developed by Erdős and A. Rényi for 5G multiuser waveform design. Furthermore, we explore using the "Best Match" algorithm, which has a polynomial time complexity, in 5G communications.**

***Index Terms*—Non-orthogonal multiple-access (NOMA), code-division multiple access (CDMA), low-density spreading signatures (LDS), sparse-code multiple-access (SCMA).**

I. Introduction

In the last decade, wireless communication services have experienced explosive growth while communication technologies have progressed generation by generation. In the previous generations spanning from 1G to 4G, the multiple access schemes were mostly characterized by orthogonal multiple access (OMA) techniques, where users are assigned orthogonal resources in either frequency, (frequency-division multiple access (FDMA)), time, (time-division multiple access (TDMA)) or code, (code-division multiple access (CDMA)). CDMA [1] was the basic technology for 3G and for some 2G (IS-95) networks. However, the multiple access scheme in 5th generation (5G) is required to support a very wide range of use cases. While 6G is in its infancy, the standardization of 5G is expected to be completed by 2020. Massive multiple-input multiple-output (MIMO), massive machine-type communication (mMTC), millimeter-wave communications, and non-orthogonal multiple access (NOMA) are four promising technologies that are expected to address the requirements of 5G wireless communications, including high spectral- and energy-efficiency, massive connectivity and low latency [2].

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Supporting a large number of users communicating over a common channel may not be readily achievable by OMA techniques due to the multiple-access interference (MAI) in rank-deficient systems, where the number of users is higher than that of the resource blocks. To meet the demand of increased bandwidth efficiency in synchronous CDMA, a CDMA concept was introduced in [3], which can support many more users for a given code length compared to traditional CDMA. A number of signature designs have been conceived [4]–[6], where low cross-correlation sequence sets are designed to minimize the overall MAI which allows more users to simultaneously access the common channel. This in turn results in increased spectral efficiency. Finding suitable spreading codes and decoding schemes for such overloaded systems is a challenging optimization problem. To address these challenges, numerous non-uniquely decodable (non-UD) [3]–[10], and UD [11]–[35] construction based code sets have been proposed. Examples of such non-UD code sets are pseudo-noise spreading (PN) [4], [7], orthogonal/orthogonal CDMA (O/O), [5], [6], PN/orthogonal CDMA (PN/O) [3], multiple-orthogonal CDMA (MO) [8], improved O/O CDMA [9]. Those codes employ two or multiple sets of orthogonal signal waveforms, which makes it possible to accommodate more users than the signature length L. As a consequence of this, a significant level of MAI exists at the output of each user's matched filter due to the non-zero cross-correlation of different signatures.

Low cross-correlation sequence sets might not be the best criterion for very high rank-deficient systems. One important criterion in such rank-deficient systems is for the code set to be uniquely decodable (UD). By definition the UD codes can be unambiguously decoded in a noiseless channel using linear recursive decoders [31]. Low-complexity linear decoders were introduced for these UD code sets using either binary $\{0, 1\}$, or antipodal $\{\pm 1\}$, or alternatively ternary $\{0, \pm 1\}$ chips in [32], [34], [36]. Due to the reason that UD code sets were originally designed for the adder channels, generally a simple noiseless detection is developed. However in practice, the

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wireless transmission channel exhibits, among other things, selective fading, multipath and the near-far problem, which leads to unequal received power among users. Consequently, if synchronization, channel equalization are compensated, low-complexity noiseless detectors can be applied to wireless channels. Alternatively, linear and non-linear detectors (e.g. match filter (MF), minimum mean square error (MMSE), message passing algorithm (MPA) etc.) are practiced for wireless channel communications.

All of these multiple access concepts were introduced in order to serve a number of excess users beyond the available resources. These multiple access schemes are characterized by non-orthogonal multiple access (NOMA) techniques [37] in 5G and beyond wireless communications. Recently, several NOMA solutions have been actively investigated [2], which can be basically divided into two main categories, namely power-domain NOMA (PDM-NOMA) and code-domain NOMA (CDM-NOMA). A few of the strong contenders of code-domain NOMA are low-density spreading aided CDMA (LDS-CDMA) [38], [39], low-density spreading assisted orthogonal frequency-division multiplexing (LDS-OFDM) [40], and sparse code multiple access (SCMA) [36], [41], pattern division multiple access (PDMA) [42] and multi-user shared access (MUSA) [43]. Besides the fact that LDS-CDMA [38], [39] is generally desired to be UD code set [44], it offers flexible resource allocation, performs better in terms of handling the MAI imposed by rank-deficient systems and has low-complexity receivers compared to conventional CDMA. The LDS-CDMA structure of code set can be represented by a factor graph, the classic MPA can be employed for its detection. LDS-CDMA, may also be appropriate for IoT communications [37] and it is also considered as a potential candidate for uplink machine-type-communications (mMTC) [37].

There has been lot of interest from academia and industry to conduct research in 5G and beyond technologies. Many countries are at a race trying to set up testbeds for 5G communication to be able to get better understanding of an end-to-end system. As a consequence, the "5G and Beyond" workshop [45], which was held in Armenia, was organized by the National Academy of Sciences, Institute for Informatics and Automation Problems, University of Ottawa Canada, National Instruments Armenia, and American University of Armenia. Therefore, the objective of this manuscript is to highlight some of the interesting research problems that where brought up during the workshop discussions. It provided us a pathway to explore solutions from other discipline such as algebraic combinatorics. We have discussed some potential research problems, which we list in the following lines:

a. We need to study the geometric structure of the received signal vector in order to develop a low-complexity decoding schemes proposed in [46], [47].
b. We may also apply novel algebraic finding of Moore-Penrose inversion and the symmetric, diagonally dominant, M-matrices (SDDM) solvers for our linear system of equations. Furthermore, we need to explore the code sets that possess SDDM property.
c. Discrete tomography constructs different matrices - with weighted rows and columns, with distinct rows, etc. [48], [49]. How can we make use of these matrices for a multiuser waveform design?
d. Random matrices in sense of Erdős-Rényi may be sparse, may have the prescribed row and/or column weights and different rows. Can these matrices be explored for NOMA waveform design?
e. For detecting the user information symbol, can we utilize the best match search algorithm, shown in Table II, or explore other advanced machine learning algorithms?

This paper is devoted to the discussion and analysis of all the potential research questions raised in a. - e. The following notations are used in this paper. All boldface lower case letters indicate column vectors and upper case letters indicate matrices, $()^T$ denotes transpose operation, sgn denotes the sign function, and $|\cdot|$ denotes complex amplitude.

II. Formulation and Foundation Development

In the following section, we describe received vector represented by all multiplexed signals in the absence of noise. Suppose that K signals contribute ± 1 information bits, the multiplexed signal vector is expressed as

$$\mathbf{y} \;=\; \mathbf{C}\mathbf{x} = \sum_{k=1}^{K} \mathbf{c}_k x_k, \tag{1}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{L-1} \\ y_L \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{k,1} & \cdots & c_{K,1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{1,L-1} & \cdots & c_{k,L-1} & \cdots & c_{K,L-1} \\ c_{1,L} & \cdots & c_{k,L} & \cdots & c_{K,L} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_K \end{bmatrix}, \tag{2}$$

where $y_l \in \mathbb{Z}$, $c_{k,l} \in \{0, \pm 1\}$, $x_k \in \{\pm 1\}$ for $1 \leq l \leq L$, $1 \leq k \leq K$, $\mathbf{y} \in \mathbb{Z}^{L\times 1}$, $\mathbf{C} \in \{0, \pm 1\}^{L\times K}$ is the code set, $\mathbf{c}_k \in \{0, \pm 1\}^{K\times 1}$ is the k-th signal signature, $k = 1, \cdots, K$, and $\mathbf{x} \in \{\pm 1\}^{K\times 1}$ is the information bit vector.

The property of the UD code set is that all possible 2^K bit-weighted sums of the $\mathbf{c}_k$ signatures are distinct in other words it is invertible $\mathbf{x}_i \neq \mathbf{x}_j \implies \mathbf{y}_i \neq \mathbf{y}_j$, where $i \neq j$. This means that we can recover $\mathbf{x}$ unambiguously (uniquely) and correctly from $\mathbf{y}$. Mathematically, it can be stated that a code set $\mathbf{C}^* \in \{0, \pm 1\}^{L\times K}$ is UD over signals $\mathbf{x} \in \{\pm 1\}^{K\times 1}$ or $\mathbf{x} \in \{0, 1\}^{K\times 1}$, if and only if, for any $\mathbf{x}_1 \neq \mathbf{x}_2$, $\mathbf{C}\mathbf{x}_1 \neq \mathbf{C}\mathbf{x}_2$ or, equivalently, $\mathbf{C}(\mathbf{x}_1 - \mathbf{x}_2) \neq \mathbf{0}_{L\times 1}$. We can rewrite the unique decodability necessary and sufficient condition as $\mathsf{Null}^\dagger(\mathbf{C}) \cap \{0, \pm 2\}^{K\times 1} = \{0\}^{K\times 1}$ or in an equivalent manner as

$$\mathsf{Null}(\mathbf{C}) \cap \{0, \pm 1\}^{K\times 1} = \{0\}^{K\times 1}. \tag{3}$$

*It is only an example that the code set alphabet is in $\{0, \pm 1\}$, in general $\mathbf{C} \in \mathbb{C}^{L\times K}$.

†$\mathsf{Null}()$ denotes the kernal or nullspace of a linear map, that is $\mathsf{Null}(\mathbf{C}) = \{\mathbf{x} | \mathbf{C}\mathbf{x} = \mathbf{0}\}$.

One obvious remark is that in noiseless channels the spreading matrix $\mathbf{C}$, should have distinct rows. This is to avoid redundancy of the received signal $\mathbf{y}$. If two rows of the spreading matrix $\mathbf{C}$ are identical then the corresponding coordinates of received signal $\mathbf{y}$ will always be equal and of course it will be considered redundant transfer of information. So the distinct row constraint for spreading matrix is mandatory.

There are many construction designs in literature for spreading matrices. One of the first objective is to achieve larger minimum Euclidean code distance. A desirable condition for design is to be UD code set. As such most of the constructions matrices can significantly outperform a randomly generated signature matrix due to a large minimum code distance property. The error-correcting code theory played a big role in designing such matrices. Another construction is based on combinatorial structures with balanced incomplete block design is studied e.g., Steiner system (STS). However, there are many other well-known studied fields that encounter construction of matrices that posses different properties [50]–[55]. We may need to revise those matrix designs and study the usability for wireless communications specifically for NOMA schemes. In Section III and IV we provide alternatives to the existing block-design approaches for constructing of spreading matrices.

III. Discrete tomography on matrices

The simplest scheme of discrete tomography is in terms of ordinary matrices [56]. We are given by row or/and column sums of a hypothetic binary $\{0,1\}$ matrix and ask about the existence of such a matrix, and then try to construct it [57]. It is also important to estimate the total number of such existing matrices for a given input. Besides the row and column values, additional qualitative conditions may be applied - horizontal and/or vertical convexity, diagonal weights, connectivity, etc. Some of these problems have polynomial complexity algorithms, but the general problem is NP complete. A number of 5G publications, designing the spreading matrix, require to be sparse, with low and equally weighted rows and columns. Additionally, they require that rows of the matrix to be distinct. H. Sahakyan [58] was the first who introduced and investigated the row distinct condition in discrete tomography. A fragment of these research is presented below.

Theorem 1. *Let $\mathbf{r} = [r, \cdots, r]^T$ and $\mathbf{s} = [s, \cdots, s]$ be the integer column and row vectors of length m and n, correspondingly, such that $r \leq n$ and $s \leq m$. Then there exists a binary matrix $\mathbf{A} \in \{0,1\}^{m \times n}$ with distinct (non-repeating) rows and the row sum vector $\mathbf{r}$ and column sum vector $\mathbf{s}$, if and only if $m \leq C_n^r$[‡] and $s \cdot n = r \cdot m$.*

Proof. **Necessity**:
Obviously, if a binary matrix $\mathbf{A}$, $a_{i,j}$ denoting the element of row i and column j, of size $m \times n$ has row sum vector $\mathbf{r}$ and column sum vector $\mathbf{s}$, then the relation $s \cdot n = r \cdot m$ obeys. If $\mathbf{A}$ consists of all distinct rows, then $m \leq C_n^r$ since there are at most C_n^r different binary vectors of size n which has r elements equal to 1.

[‡]In combinatorics $C_n^r = \binom{n}{r}$ is usually read as n choose r.

TABLE I
Algorithm A

Algorithm A

Input: $\mathbf{d} = [d_1, \ldots, d_t, 0, \ldots, 0, d_k, \ldots, d_n]$
1: **while** $t \geq 1$ AND $k \leq n$
2: $\quad i \leftarrow t$, $j \leftarrow k$
3: $\quad$ **while** $d_i > 0$ AND $d_j < 0$
4: $\quad\quad d_i \leftarrow d_i - 1$, $d_j \leftarrow d_j + 1$
5: $\quad\quad$ **if** $d_i = 0$
6: $\quad\quad\quad t \leftarrow t - 1$
7: $\quad\quad$ **else if** $d_j = 0$
8: $\quad\quad\quad k \leftarrow k + 1$
Output: $\mathbf{d}$

Sufficiency:
Suppose that $m \leq C_n^r$ and $s \cdot n = r \cdot m$. We can see that $m \leq C_n^r$ implies that there exists a binary matrix of size $m \times n$ with distinct rows and with the row sum vector $\mathbf{r}$ (since we can take m different binary vectors of size n that has r elements equal to 1, and compose a matrix). Note that there are $C_{C_n^r}^m$ such matrices, and denote their class by $\mathcal{M}_m^r(n)$. We have to prove that $\mathcal{M}_m^r(n)$ contains a matrix with the column sum vector $\mathbf{s}^* = [\frac{r \cdot m}{n}, \ldots, \frac{r \cdot m}{n})$. Let $\mathbf{A}$ be an arbitrary matrix from $\mathcal{M}_m^r(n)$ with the row sum vector $\mathbf{r} = [r, \cdots, r]^T$ and let $\mathbf{s} = [s_1, \cdots, s_n]$ be its column sum vector. We intend to transform matrix $\mathbf{A}$ into a matrix $\mathbf{A}^*$ of $\mathcal{M}_m^r(n)$ which has column sum vector $\mathbf{s}^*$. First we introduce an algorithm, shown in Table I, which transforms $\mathbf{s} = [s_1, \cdots, s_n]$ to $\mathbf{s}^*$, and then we will show that the algorithm can be realized on the matrices of $\mathcal{M}_m^r(n)$. Without loss of generality, we can assume that the elements $s_i \in \mathbb{N}$ for $1 \leq i \leq n$ and are arranged in the decreasing order, $s_1 \geq \cdots \geq s_{n-1} \geq s_n$. Each s_i can either be greater than, or less than, or equal to $(r \cdot m)/n$. Suppose $s_1 > \frac{r \cdot m}{n}, s_2 > \frac{r \cdot m}{n}, \ldots, s_t > \frac{r \cdot m}{n}, s_{t+1} = \frac{r \cdot m}{n}, s_{t+2} = \frac{r \cdot m}{n}, \ldots, s_{k-1} = \frac{r \cdot m}{n}, s_k < s, s_{k+1} < s, \ldots, s_n < s$. Then, we compose a vector $\mathbf{d} = [d_1, \ldots, d_n]$ in the following way $d_i = s_i - \frac{r \cdot m}{n}$ for $i = 1, \ldots, n$. Therefore, $\sum_{i=1}^{t} d_i = -\sum_{i=k}^{n} d_i$ with $d_{t+1} = \cdots = d_{k-1} = 0$.

At the end of the algorithm in Table I, all elements of the vector $\mathbf{d}$, results in $d_i = 0$ for $1 \leq i \leq n$. Now looking at line 4 of algorithm in Table I, we prove that those can be achieved by transforming the initial matrix $\mathbf{A}$ in such a way that the i-th element of its column sum vector will decrease by 1, and the j-th element will increase by 1. Therefore, $d_i \leftarrow d_i - 1$, $d_j \leftarrow d_j + 1$ imply that $s_i - s_j \geq 2$. Then, it follows that there exist two rows, let say k_1-th and k_2-th such that $a_{k_1,i} = 1$ and $a_{k_2,i} = 1$, but $a_{k_1 j} = 0$ and $a_{k_2,j} = 0$. Replacing 1s with 0s in $a_{k_1,i} = 0$ and 0s with 1s in $a_{k_1 j} = 1$ results in a matrix with $s_i - 1$ on the i-th column, and $s_j + 1$ on the j-th column. Therefore, this completes the proof.

□

IV. Random sets and matrices

Random graphs became popular after the initial works of P. Erdős and A. Rényi [59]. This is not one idea but a

broad theory now, and our aim is to understand the picture of random matrix generation in a probabilistic environment. And of course our target, as in precious section is the alternative way of constructing of low and equal line weight matrices with different rows.

Consider a random matrix $\mathbf{A} \in \{0,1\}^{m\times n}$, where the column vectors $\mathbf{a}_j$ for $j = 1,\ldots,n$ independently and identically attain values 1 and 0 with nonzero probabilities p_j and $q_j = 1 - p_j$. The number of possible generated binary matrices is 2^{mn} and the probability of a matrix is tightly related with the number of 1 values in its columns. Given a matrix, and column sums $s_1, s_2, \ldots, s_n$ then its probability is equal to $\prod_{j=1}^{n} p_j^{s_i} q_j^{m-s_j}$. The model is expressed as a process, where m vertices are i.i.d. as it is defined above are dropped into the hypercube, $\mathcal{E}_2^{n\times 1}$, where $\mathcal{E}_m = \{0,1,\ldots,m-1\}$. There might be repeated hypercube vertices however in the case of no row repetitions vertices compose of m-subset of the $\mathcal{E}_2^{n\times 1}$. The indicator of the existence of such distinct row matrices is related the nonzero probability - in the given model. We suppose that the probabilities, $p_j = s_j/m$ and intend to analyze asymptotically, when $n, m \to \infty$, the following:

(i) What is the probability that column sums $s_1, s_2, \ldots, s_n$ tend to 1.

(ii) What is the probability that all the rows are distinct tends to 1.

Consider an arbitrary column j and let m_j and d_j be the average and dispersion[§] of the random weight of column j. Probability that the sum column is equal to t of a random matrix, equals $C_m^t p_j^{s_i} q_j^{m-s_j}$. As a result $m_j = \sum_{t=0}^{m} t C_m^t p_j^t q_j^{m-t} = mp_j$ which is quite obvious. Now, the overall average sums by the set of coordinates/columns will be equal to $mp_1, mp_2, \ldots, mp_n$. And since $p_j = \frac{s_j}{m}$ for $j = 1,\ldots,n$, the average sum vector equals to $s_1, s_2, \ldots, s_n$. In a similar way, the expected number of different pairs of rows in the one-column model can be calculated as follows,

$$\begin{aligned} m_j &= \sum_{t=0}^{m} t(m-t) C_m^t p_j^t q_j^{m-t} \qquad (4)\\ &= \sum_{t=1}^{m-1} t(m-t) C_m^t p_j^t q_j^{m-t} \\ &= m(m-1)p_j q_j \sum_{t=0}^{m-2} C_{m-2}^t p_j^{t-1} q_j^{m-2-t} \\ &= m(m-1)p_j q_j. \end{aligned}$$

The additional use of dispersion's in this model brings more points. Combined with the Chebyshev inequality this gives intervals around the values s_j with a property that the random sums belong to these intervals with a strongly positive probability. This is our result for the point (i). The domain described by the above intervals is a rectangular area in the space of all sum vectors, $\mathcal{E}_m^{n\times 1}$, and the achieved property insists that there exist a proper random sum vector that belongs to the indicated rectangular area. Setting $s_1, s_2, \ldots, s_n$ arbitrarily, we receive corresponding rectangular area of different size and probability (it can be also empty). Unless attractive, in this form the property is not yet useful, since we do not know if the rows of random matrix are different in this case. Next we look into (ii) by considering a pair of random rows. The probability that particular j-th coordinates of this pair are identical is evidently $2p_j(1-p_j)$. The probability that the entire rows are identical equals $\prod_{j=1}^{n} 2p_j(1-p_j)$ and the corresponding probability that this rows are different will be $1 - \prod_{j=1}^{n} 2p_j(1-p_j)$. For example, when $p_j = q_j = 1/2$ then we receive $\alpha = 1 - \frac{1}{2^n}$. Let us consider m elements dropped onto the $\mathcal{E}_2^{n\times 1}$. The probability that all pairs of these m elements will appear different is majored by $\beta = (1 - \prod_{j=1}^{n} 2p_j(1-p_j))^{m^2}$ and $\gamma = (1 - \frac{1}{2^n})^{m^2}$ is the value in particular case $p_j = 1/2$. The m^2 is used as a raw estimate for number of pairs, C_m^2. For growing n, and relatively small m, $\gamma \sim e^{-\frac{m^2}{2^n}}$. We suppose that γ is non zero then derive from this that there must exist a matrix with different rows. In general case, there is a deviation from the $p_j = 1/2$ which can be maintained inserting a deviation coefficient δ so that $\beta = \gamma\cdot\delta$. Then, it is to determine the proper value of m so that the probability β is non zero that derives to the existence of a row-different matrix. Combining (i) and (ii) for given $s_1, s_2, \ldots, s_n$ we receive conditions and a rectangular area of $\mathcal{E}_m^{n\times 1}$ around $s_1, s_2, \ldots, s_n$, that contains a random row-distinct matrix. This probabilistic result can be extended inserting into the consideration the dispersion values. Considerations above intend to get an additional knowledge about the row-different matrices using the probabilistic theory of combinatorics [51], [59]–[61]. The objective is reasonable because the pure combinatorial approach is not able at the moment to give an efficient description of the column weighted row-distinct matrices. The probabilistic method gives a knowledge on random subsets, which might be useful as a complementary knowledge about a different object or a situation.

[§]A measure of statistical dispersion is a nonnegative real number that is zero if all the data are the same and increases as the data become more diverse.

V. Multiuser detectors

In order to allow K simultaneously transmitting users to share common resources, we will focus on the design of a set of K complex-valued spreading sequences $\mathbf{c}_k \in \mathbb{C}^{L\times 1}$, for $k = 1,\ldots,K$. In particular, we will focus on the case $L < K$, commonly referred to as overloaded systems. We investigate complex baseband received vector in the presence of noise, which can be expressed as

$$\begin{aligned} \mathbf{y} &= A\mathbf{C}\mathbf{x} + \mathbf{n} \qquad (5)\\ &= \sum_{k=1}^{K} A\mathbf{c}_k x_k + \mathbf{n}, \qquad (6) \end{aligned}$$

where A denotes the amplitude, $\mathbf{x} \in \mathbb{X}^{K\times 1}$ is the user's information symbols taken from some finite, complex-valued signal constellation $\mathbb{X}$ and $\mathbf{n}$ is the additive white Gaussian noise (AWGN) vector with a variance of σ^2.

The detection of the information symbols of user k can be achieved via linear detectors such as correlation demodulator, MF demodulator and MMSE (or, equivalently, minimum total square correlation (min-TSC) filtering, maximum signal-to-noise ratio (max-SINR) filtering). Although these filtering exhibits low-complexity multiuser detection, their performance is suboptimum. The widely recognized optimum decision rule based on the observation vector $\mathbf{y}$ is such that the probability of a correct decision is maximized. We consider a decision rule based on the computation of the *posterior probabilities* defined as

$$P(\mathbf{x}|\mathbf{y}), \quad k = 1, \ldots, K, \tag{7}$$

where $\mathbf{x}$ is unknown transmitted symbol vector and $\mathbf{y}$ is observed received vector. The decision criterion is based on selecting the signal $\mathbf{x}$ such that it maximized the *a posteriori probability*, $P(\mathbf{x}|\mathbf{y})$, in turn it maximizes the probability of correct decision and, hence, minimizes the probability of error. This decision criterion is called the maximum *a posteriori probability* (MAP) [62] criterion.

$$\widehat{\mathbf{x}} = \underset{\mathbf{x} \in \mathbb{X}^{K \times 1}}{\operatorname{argmax}} P(\mathbf{x}|\mathbf{y}). \tag{8}$$

Using Bayes' rule the posterior probabilities may be expressed as

$$P(\mathbf{x}|\mathbf{y}) = \frac{P(\mathbf{y}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{y})}, \tag{9}$$

where $P(\mathbf{y}|\mathbf{x})$ is the conditional probability density function of the observed vector given $\mathbf{x}$, and $P(\mathbf{x})$ is the *a priori probability* of the signal being transmitted. If we further assume that transmit vectors $\mathbf{x}$ are equally probable a priori, i.e., $P(\mathbf{x}_i) = 1/|\mathbb{X}|^K$ for all $1 \leq i \leq |\mathbb{X}|^K$. Consequently, the decision rule based the signal that maximizes $P(\mathbf{x}|\mathbf{y})$ is equivalent to finding the signal that maximizes $P(\mathbf{y}|\mathbf{x})$, which is called the maximum-liklihood (ML) criterion

$$\widehat{\mathbf{x}} = \underset{\mathbf{x} \in \mathbb{X}^{K \times 1}}{\operatorname{argmax}} P(\mathbf{y}|\mathbf{x}). \tag{10}$$

In the case of an AWGN channel (5), the ML solution is given by

$$\widehat{\mathbf{x}} = \underset{\mathbf{x} \in \mathbb{X}^{K \times 1}}{\operatorname{argmin}} ||\mathbf{y} - A\mathbf{C}\mathbf{x}||^2. \tag{11}$$

It is widely recognized that obtaining the ML solution is generally NP-hard [63]. Therefore, we explore polynomial time complexity algorithm in the next section.

VI. Best match search model

In this section, we try to approach the multiuser detection problem from the error-correcting coding viewpoint. We consider the received vector in noiseless case (1), $\mathbf{y} \in \mathbb{Z}^{L \times 1}$. Definitely, for a given multiuser matrix $\mathbf{C}$, all the generated integer vectors, $\mathbf{y}$, belong to a 2^K-subset, $\mathcal{Z}$, in space $\mathbb{Z}^{L \times 1}$. In other words $\mathcal{Z} \subset \mathbb{Z}^{L \times 1}$. Taking the $\mathcal{Z}$, a 2^K-subset to be a codebook from coding theory we can apply minimum distance decoding algorithm to solve (11) as follows,

$$\widehat{\mathbf{y}} = \underset{\mathbf{y}' \in \mathcal{Z}}{\operatorname{argmin}} \mathsf{d}_{\mathsf{H}}(\mathbf{y}, \mathbf{y}'), \tag{12}$$

where $\mathsf{d}_{\mathsf{H}}(\cdot)$ denotes Hamming distance. Without loss of generality, the "Best Match" algorithm that works on space $\mathcal{E}_2^{L \times 1}$ can be applied on space $\mathcal{Z}$ as well. Let $\mathcal{F}$ be a finite set of some binary words of length L. Let $\mathcal{F}(\mathbf{x})$ denote the set of all the words from $\mathcal{F}$ having the (same) minimal possible distance from $\mathbf{x}$ (in the simplest case the Hamming distance is applied). Best match search algorithms generate the best match set $\mathcal{F}(\mathbf{x})$ by the given $\mathcal{F}$ and $\mathbf{x}$. The trivial algorithm may pass the set $\mathcal{F}$ placed in the memory of a machine each time, for the given $\mathbf{x}$. We need to optimize algorithm performance by a better management of $\mathcal{F}$ in the memory. P. Elias was the first to suggest the use of the block balanced partition of $\mathcal{F}$. In fact, partition is applied over the entire L-cube and partition of $\mathcal{F}$ is a consequence of this. The initial method [46], [64] was restricted to the use of limited set of possible perfect codes that narrow the special constructions of these partitions. It is well known that the only nontrivial classes of binary perfect codes are the Hamming and Golay codes [49], [50]. We use the geometric interpretation of perfect codes, when the code based none intersecting system of spheres is given. Linear codes additionally allow optimizing of addressing issues in considered models. The codeword centered spheres play the role of blocks. So, let us suppose that the basic set $\mathcal{E}_2^{L \times 1}$ of all binary words of length L is divided into the blocks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$. Accordingly, $\mathcal{L}_i = \mathcal{F} \cap \mathcal{B}_i$, $i = 1, 2, \ldots, m$ are the sets of elements of $\mathcal{F}$ belonging to these blocks. They are saved in memory as separate lists by addresses $\mathsf{h}^{\P}(\mathcal{B}_i)$, common for all elements of each such block of $\mathcal{E}_2^{L \times 1}$. The idea of this construction is transparent: using a dynamic programming branch and bound type algorithm and a partitioning of the space $\mathcal{E}_2^{L \times 1}$ into the geometrically compact blocks, the algorithm may need less information readings to set up $\mathcal{F}(\mathbf{x})$ by an input $\mathbf{x}$. To achieve this result we have to divide $\mathcal{E}_2^{L \times 1}$ into the disjoint blocks and splitting into the spheres by a perfect code is considered to be a good choice.

Let us summarize the main points of structuring $\mathcal{E}_2^{L \times 1}$ for the best match search: $\mathcal{E}_2^{L \times 1}$ is divided into the simple blocks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$. Blocks are equal in their sizes, and the lists of elements in blocks $\mathcal{L}_i = \mathcal{F} \cap \mathcal{B}_i$, $i = 1, 2, \ldots, m$ practically have comparable sizes. Blocks are non-intersecting. Given any arbitrary $\mathbf{x}$, blocks can be quickly and simply arranged by their distances from $\mathbf{x}$. 4 bold points condition the effectiveness of the algorithm. The simplest partition as we mentioned, is by a Hamming code. But these codes exist for dimensions $L = 2^q - 1$ only. And the block size by Hamming codes is very much limited: $L+1$. The same time ranging blocks by the input vector $\mathbf{x}$ is very quick and ideally simple. Other codes such as Golay codes and non-linear codes or codes in other distances do minimal help. It is also to take into account the

$^{\P}\mathsf{h}(\cdot)$ denotes a hash function.

quasi-perfect and nearly perfect codes, which provide more constructions for approximating the problem.

A. Block Construction by Quasi-perfect Codes

An (L, t)-quasi-perfect code is a code for which the spheres of radius t around the code words are disjoint, and every binary L-vector is at most $t+1$ distance from some code word. A subclass of quasi-perfect codes is nearly perfect codes. A few nearly perfect code sets are known. For $t = 1$ there exists a nearly perfect code for any $L = 2^q - 2$. For $t = 2$ there exist a nearly perfect code for $L = 4^q - 1$. It is also known that no other nearly perfect codes exist. For any nearly perfect code: vectors with a greater than t distance from any code word is at distance $t+1$ from exactly $\lfloor L/(t+1) \rfloor$ other code words; and vectors of distance t from some code word are at distance $t+1$ from exactly $\lfloor (L-t)/(t+1) \rfloor$ other code words [65]–[67]. In Table II, the input $\mathcal{F} \neq \emptyset$, $\delta \in \mathbb{N}$ denotes the current best

TABLE II
ELIAS ALGORITHM

Elias

Input: $\mathbf{x}$, $\mathcal{F}$
1: $\delta \leftarrow \infty$, $\mathcal{S} \leftarrow \emptyset$, $j \leftarrow -1$
2: **while** $j < \delta$
3: $j \leftarrow j + 1$
4: **if** $s(j) \neq 0$
5: **for** $i \leftarrow 0$ to $i < s(j)$
6: **if** $\mathcal{L}_{j_i} \neq \emptyset$
7: **if** $\delta \leq d(\mathbf{x}, \mathcal{L}_{j_i})$
8: $\mathcal{S} \leftarrow \mathcal{S} \cup (\mathcal{O}_\delta^n(\mathbf{x}) \cap \mathcal{L}_{j_i})$
9: **else** $\mathcal{S} = \mathcal{O}_\delta^n(\mathbf{x}) \cap \mathcal{L}_{j_i}$
10: $\delta \leftarrow d(\mathbf{x}, \mathcal{L}_{j_i})$
Output: $\mathcal{S}$

match distance, $\mathcal{S}$ is the current set of vectors of $\mathcal{F}$ located at distance δ from $\mathbf{x}$, $j \in \mathbb{N}$ denotes current distance of blocks under consideration from $\mathbf{x}$, $s(j)$ is the number of blocks in distance j from $\mathbf{x}$, $\mathcal{L}_{j_i}$ is the i-th list with j distance block and $\mathcal{O}_\delta^n(\mathbf{x})$ is δ neighborhood of $\mathbf{x}$. Efficiency of Elias algorithm can be summarized as follows,

- As the block structure perfect codes can be used, e.g. Golay code with $n = 23$, $d = 7$.
- $\mathcal{F}_q = \{\mathbf{y} \in \mathcal{E}_2^{L\times 1} | d(\mathbf{x}, \mathbf{y}) = d(\mathbf{q}, \mathcal{F})\}$, where $d(\mathbf{q}, \mathcal{F}) = \operatorname{argmin}_{\mathbf{z}\in\mathcal{F}} d(\mathbf{x}, \mathbf{z})$.
- Let, all $\mathbf{q} \in \mathcal{E}_2^{L\times 1}$ are equally likely, and $\mathcal{F}$ is a random set of vertices of $\mathcal{E}_2^{L\times 1}$ with membership probability p. Experiments for reasonable values of p show that in average algorithm analyses only 0.9% of $\mathcal{F}$ for $\mathcal{F}_q$.
- Hypotheses that optimal block structures are discrete isoperimetry problem (DIP) solutions [46], [47] belongs to R. Rivest.

B. The Isoperimetry Block Optimality

To estimate the average work of the "Best Match" algorithms over all queries $\mathbf{x}$ and files $\mathcal{F}$, and assume that each query $\mathbf{x} \in \mathcal{E}_2^{L\times 1}$ is equally likely with a probability p so that any point $\mathbf{x} \in \mathcal{E}_2^{L\times 1}$ appear in $\mathcal{F}$ independently of any other point by this probability. Denote by $\Psi(\mathcal{B}_i)$ the probability that the list $\mathcal{L}_i$ will be treated by the algorithm. Then the DIP optimality proof found in [66] can be summarized as follows.

Proof. Let $\mathcal{C}$ and $\mathcal{B}$ are arbitrary block structures. Consider and compare probabilities that these blocks are analyzed for constructing the set $\mathcal{F}(\mathbf{q})$:

$$\Psi(\mathcal{C}) - \Psi(\mathcal{B}) = 2^{-n}(|\mathcal{C}^{(m)}| - |\mathcal{B}^{(m)}|)\Theta(n, m-1) + 2^{-n} \sum_{m<\delta\leq n} (|\mathcal{C}^{(\delta)}| - |\mathcal{B}^{(\delta)}|)\Theta(n, m-1) \geq 2^{-n}(|\mathcal{C}^{(m)}| - |\mathcal{B}^{(m)}|)(\Theta(n, m-1) - \Theta(n, m)) \geq 0. \quad (13)$$

Let $\mathcal{C}$ is an arbitrary block and $\mathcal{B}$ is a DIP block. Let $\mathcal{C}^{(\delta)}$ and $\mathcal{B}^{(\delta)}$ are vertices of distance δ from these blocks. $\Theta(n, m)$ is probability that a sphere of radius m is empty (of $\mathcal{F}$). m is the position/index, that $|\mathcal{C}^{(i)}| = |\mathcal{B}^{(i)}|$ for $i = (1, m-1)$ but $|\mathcal{C}^{(m)}| > |\mathcal{B}^{(m)}|$.

□

VII. CONCLUSION

In this paper, we presented some of the interesting research questions raised during the "5G and Beyond" workshop discussions held in Armenia. We proposed to explore alternative solutions in other discipline such as algebraic combinatorics. Construction of matrices that possess certain properties in discrete tomography can be exploited for the 5th generation (5G) multiuser waveform design. Moreover, we proposed to investigate thoroughly the random graph theory developed by Erdős and A. Rényi for 5G multiuser code design. We explored "Best Match" algorithm, which has polynomial time complexity for potentially using in 5G communications.

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