

GENERATING FUNCTIONS OF CHEBYSHEV-LIKE POLYNOMIALS

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ABSTRACT. In this short note, we give simple proofs of several results and conjectures formulated by Stolarsky and Tran concerning generating functions of some families of Chebyshev-like polynomials.

1. INTRODUCTION

We recall the definition of discriminant. Let $P(x)$ be a polynomial of degree m with the leading coefficient a and m roots x_1, x_2, \dots, x_m . Then the discriminant of $P(x)$ is

$$\Delta_x P(x) = a^{2m-1} \prod_{i \neq j} (x_i - x_j)^2.$$

Stolarsky [4] observed that the discriminant of the product of two polynomials $K(x, q)$, whose distinct roots satisfy $|x| \neq 1$ unless $a \leq q \leq b$, and $f_m(x)$ whose distinct roots lie on $|x| = 1$ will be a polynomial in q whose roots are all in $[a, b]$. In particular if we let

$$K(x, q) = K_s(x, q) = (1+x)^{2s} + qx^s$$

and

$$f_m(x) = (x^{2m+1} - 1)/(x - 1)$$

then the discriminant of their product is [5]

$$\Delta_x(K_s f_m) = C_m^{(s)} q^{2s-1} (q + 2^{2s}) H_m^{(s)}(q)^4$$

where $C_m^{(s)}$ is a constant given by

$$C_m^{(s)} = (-1)^m (2m+1)^{2m-1} s^{2s}$$

and

$$H_m^{(s)}(x) = \prod_{k=1}^m \left(x + 4^s \cos^{2s} \left(\frac{k\pi}{2m+1} \right) \right).$$

For $s = 1$, the polynomials $H_m^{(s)}(x)$ are related to the classical Chebyshev polynomials of the second kind $U_n(x)$. From the definition

$$U_n(x) = 2^n \cdot \prod_{k=1}^n \left(x - \cos \frac{k\pi}{n+1} \right),$$

it follows that $U_{2m}(x) = (-1)^m \cdot H_m^{(1)}(-4x^2)$. From there a simple derivation gives the generating series of the polynomials $H_m^{(1)}(x)$.

Proposition 1. *The generating series of the sequence $H_n^{(1)}(x)$ is rational:*

$$\sum_{m \geq 0} H_m^{(1)}(x)t^m = \frac{1-t}{(1-t)^2 - xt}.$$

Proof. Starting from the classical rational generating series

$$(1) \quad \sum_{n \geq 0} U_n(x)t^n = \frac{1}{1-2tx+t^2},$$

a generating series for the even part is readily obtained:

$$\sum_{m \geq 0} U_{2m}(x)t^{2m} = \frac{1}{2} \left(\frac{1}{1-2tx+t^2} + \frac{1}{1+2tx+t^2} \right) = \frac{1+t^2}{(1+t^2)^2 - 4x^2t^2},$$

and thus

$$\sum_{m \geq 0} H_m^{(1)}(-4x^2)t^m = \sum_{m \geq 0} U_{2m}(x)(-t)^m = \frac{1-t}{(1-t)^2 + 4x^2t},$$

from which the conclusion follows. \square

In [4], conjectures were given concerning the generating functions of $H_m^{(s)}(x)$. In [5], Proposition 1 was proved in a different way, and the generating function for the case $s = 2$ was shown to be rational. We generalize these results by proving.

Theorem 1. *For any $s \geq 1$, the generating series $F_s(t, x) = \sum_{m \geq 0} H_m^{(s)}(x)t^m$ is rational. Its denominator has degree at most 2^s in t , and its numerator has degree at most $2^s - 1$ in t .*

Moreover, we give an algorithm that computes these rational functions explicitly. In Section 2, we give the proof of this theorem. In Section 3, we comment on the computational aspects. We conclude with a few further observations in Section 4.

2. RATIONAL GENERATING SERIES

For convenience, we work with the polynomial $G_m^{(s)}(x) = (-1)^m H_m^{(s)}(-x)$. It is monic, with roots the s -th powers of the roots of $G_m^{(1)}(x) = (-1)^m H_m^{(1)}(-x)$. To prove the theorem, it is clearly sufficient to show that the generating series $\sum_{m \geq 0} G_m^{(s)}(x)t^m$ is rational.

2.1. Roots. Letting ε_s be a primitive s -th root of unity, we have

$$G_m^{(s)}(x^s) = \prod_{G_m^{(1)}(\alpha)=0} (x^s - \alpha^s) = \prod_{G_m^{(1)}(\alpha)=0} \prod_{j=0}^{s-1} (x - \alpha/\varepsilon_s^j),$$

which equals

$$\prod_{j=0}^{s-1} \prod_{G_m^{(1)}(\alpha)=0} (x - \alpha/\varepsilon_s^j) = \prod_{j=0}^{s-1} G_m^{(1)}(\varepsilon_s^j x) \cdot \prod_{j=0}^{s-1} \varepsilon_s^{-jm} = (-1)^{m(s-1)} \cdot \prod_{j=0}^{s-1} G_m^{(1)}(\varepsilon_s^j x).$$

2.2. Polynomials, Recurrences and Hadamard Products. By Proposition 1, for any $j \geq 0$, the generating series of $G_m^{(1)}(\varepsilon_s^j x)$ is rational. Equivalently, the sequence of polynomials $G_m^{(1)}(\varepsilon_s^j x)$ satisfies a linear recurrence with coefficients that do not depend on m . The characteristic polynomial of this recurrence is the reciprocal of the denominator of the generating series. Let $\alpha_1(x), \alpha_2(x)$ denote the roots of this denominator for $j = 0$. The product of solutions of linear recurrences with constant coefficients satisfies a linear recurrence with constant coefficients again (see, e.g., [6, §2.4]). The generating series $\sum_{m \geq 0} G_m^{(s)}(x^s)t^m$ is therefore rational. It is called the *Hadamard product* of the generating series of the $G_m^{(1)}(\varepsilon_s^j x)$, $j = 1, \dots, s-1$. By [2, §2, Ex. 5], the reciprocal of its denominator is the characteristic polynomial

$$P_s(t, x) = \prod_{1 \leq i_1, \dots, i_s \leq 2} \left(t - \alpha_{i_1}(x) \alpha_{i_2}(\varepsilon_s x) \cdots \alpha_{i_s}(\varepsilon_s^{s-1} x) \right).$$

We prove that the polynomial $P_s(t, x)$ belongs to $\mathbb{Q}[x^s, t]$ by showing that all the (Newton) powersums of the roots of $P_s(t, x)$ belong to $\mathbb{Q}[x^s]$. For any $\ell \in \mathbb{N}$, the ℓ -th powersum is equal to the product

$$(2) \quad (\alpha_1^\ell(x) + \alpha_2^\ell(x)) (\alpha_1^\ell(\varepsilon_s x) + \alpha_2^\ell(\varepsilon_s x)) \cdots (\alpha_1^\ell(\varepsilon_s^{s-1} x) + \alpha_2^\ell(\varepsilon_s^{s-1} x)),$$

and thus has the form $T_\ell(x) = Q_\ell(x)Q_\ell(\varepsilon_s x) \cdots Q_\ell(\varepsilon_s^{s-1} x)$ for some polynomial $Q_\ell(x) \in \mathbb{Q}[x]$. The polynomial $T_\ell(x)$ being left unchanged under replacing x by $\varepsilon_s x$, it belongs to $\mathbb{Q}[x^s]$, and so do all the coefficients of P_s .

In conclusion, $G_m^{(s)}(x^s)$ satisfies a recurrence with coefficients that are polynomials in $\mathbb{Q}[x^s]$, thus the series $\sum_{m \geq 0} G_m^{(s)}(x^s)t^m$ is rational and belongs to $\mathbb{Q}(x^s, t)$, and thus $F_s(t, x^s)$ belongs to $\mathbb{Q}(t, x^s)$. The assertion on the degree in t of (the denominator and numerator of) $F_s(t, x)$ follows from the form of $P_s(t, x)$. This concludes the proof of Theorem 1.

3. ALGORITHM

The proof of Theorem 1 is actually effective, and therefore it can be used to generate, for specific values of s , the corresponding rational function F_s , in a systematic and unified way.

3.1. A new proof for the case $s = 2$. We illustrate these ideas in the simplest case $s = 2$, for which the computations can be done by hand.

We are interested in the generating series of $H_m^{(2)}(-x^2) = G_m^{(1)}(x)G_m^{(1)}(-x)$ from which that of the sequence $H_m^{(2)}(x)$ is easily deduced.

From its generating function, we deduce that the sequence $G_m^{(1)}(x)$ satisfies the second order recurrence

$$(3) \quad G_{m+2}^{(1)}(x) + (2-x)G_{m+1}^{(1)}(x) + G_m^{(1)}(x) = 0, \quad G_0(x) = 1, \quad G_1(x) = x - 1.$$

A recurrence satisfied by the product $k_m(x) = G_m^{(1)}(x)G_m^{(1)}(-x)$ can then be obtained from the identities

$$\begin{aligned} k_m(x) &= G_m^{(1)}(x)G_m^{(1)}(-x), \\ k_{m+1}(x) &= G_{m+1}^{(1)}(x)G_{m+1}^{(1)}(-x), \\ k_{m+2}(x) &= G_{m+2}^{(1)}(x)G_{m+2}^{(1)}(-x) \\ &= (4-x^2)G_{m+1}^{(1)}(x)G_{m+1}^{(1)}(-x) + (2+x)G_m^{(1)}(x)G_{m+1}^{(1)}(-x) \\ &\quad + (2-x)G_m^{(1)}(-x)G_{m+1}^{(1)}(x) + G_m^{(1)}(x)G_m^{(1)}(-x), \\ k_{m+3}(x) &= \dots \end{aligned}$$

where the right-hand sides all reduce to linear combinations of the 4 sequences $\{G_{m+1}^{(1)}(x)G_{m+1}^{(1)}(-x), G_m^{(1)}(x)G_{m+1}^{(1)}(-x), G_m^{(1)}(-x)G_{m+1}^{(1)}(x), G_m^{(1)}(x)G_m^{(1)}(-x)\}$. It follows by elimination that the sequence $k_m(x)$ satisfies the fourth order recurrence

$$k_{m+4} + (x^2 - 4)k_{m+3} + (2x^2 + 6)k_{m+2} + (x^2 - 4)k_{m+1} + k_m = 0,$$

with initial conditions

$$k_0 = 1, k_1 = 1 - x^2, k_2 = 1 - 7x^2 + x^4, k_3 = 1 - 26x^2 + 13x^4 - x^6.$$

Equivalently, the generating series $\sum_{m \geq 0} G_m(x)G_m(-x)t^m$, that is, the generating series of $H_m^{(2)}(-x^2)$, equals

$$(t^3 + 3t^2 - 3t + 1)/(t^4 + (x^2 - 4)t^3 + (2x^2 + 6)t^2 + (x^2 - 4)t + 1).$$

Finally, we have just proven that the generating series of the sequence of polynomials $H_m^{(2)}(x)$ is

$$(4) \quad \sum_{m=0}^{\infty} H_m^{(2)}(x)t^m = \frac{(1-t)^3}{(t-1)^4 - xt(t+1)^2}.$$

This provides a simple alternative proof of [5, Proposition 4.1].

3.2. General Algorithm. The elimination method outlined above applies to linear recurrences that can even have *polynomial* coefficients. It is implemented in the Maple package `gfun` [3]. This makes it possible to write a first implementation that produces the generating series of the $H_m^{(s)}(x)$ for small s .

The computation can be made more efficient by using algorithms specific to linear recurrences with constant coefficients [1]. The central idea is to recover the denominator from the (Newton) powersums of its roots, themselves obtained from the powersums of α_1 and α_2 . Using resultants, computations in algebraic extensions can be avoided. This is summarized in the following algorithm.

Input: an integer $s \geq 2$.

Output: the rational function $F_s(t, x)$.

- (1) Compute the powersums $Q_\ell(x) := \alpha_1(x)^\ell + \alpha_2(x)^\ell \in \mathbb{Q}[x]$ for $\ell = 0, \dots, 2^s$;
- (2) Compute the powersums $T_\ell(x)$ from Eq. (2) as $T_\ell(x) = \text{Res}_y(Q_\ell(y), x^s - y^s)$ for $\ell = 0, \dots, 2^s$;
- (3) Recover the denominator $D_s(x, t)$ of F_s from these powersums;
- (4) Compute $G_m^{(1)}(x)$ for $m = 0, \dots, 2^s - 1$ using the 2nd order recurrence (3);
- (5) Compute $G_m^{(s)}(x) = \text{Res}_y(G_m^{(1)}(y), x - y^s)$ for $m = 0, \dots, 2^s - 1$;

- (6) The numerator of F_s is $N_s(x, t) := D_s(x, t) \times \sum_{m=0}^{2^s-1} G_m^{(s)}(-x)(-t)^m \pmod{x^{2^s}}$;
 (7) **Return** the rational function $F_s = N_s(x, t)/D_s(x, t)$.

We give the complete Maple code in an Appendix.

3.3. Special cases. For $s = 2$, the computation recovers (4). For $s = 3$, it takes less than one hundredth of a second of computation on a personal laptop to prove the following result

$$\sum_{m \geq 0} H_m^{(3)}(x)t^m = \frac{(1-t)((t-1)^6 - xt^2(t+3)(3t+1))}{x^2t^4 - xt(t^4 + 14t^3 + 34t^2 + 14t + 1)(t-1)^2 + (t-1)^8},$$

that was conjectured by Stolarsky in [4].

Similarly, less than two hundredth of a second of computation is enough to discover and prove the following new result

Proposition 2. *The generating function $\sum_{m \geq 0} H_m^{(4)}(x)t^m$ is equal to*

$$\frac{(t-1)(x^2t^4A(t) - 2xt^2B(t)(t-1)^6 + (t-1)^{14})}{x^3t^5(t+1)^2(t-1)^4 + x^2t^3C(t) + xt(t-1)^8D(t) - (t-1)^{16}},$$

where

$$\begin{aligned} A(t) &= 9t^6 - 46t^5 - 89t^4 - 260t^3 - 89t^2 - 46t + 9, \\ B(t) &= 11t^4 + 128t^3 + 266t^2 + 128t + 11, \\ C(t) &= 2t^{10} - 13t^9 + 226t^8 - 300t^7 - 676t^6 - 2574t^5 - 676t^4 - 300t^3 + 226t^2 - 13t + 2, \\ D(t) &= t^6 + 60t^5 + 519t^4 + 1016t^3 + 519t^2 + 60t + 1. \end{aligned}$$

The code given in the appendix makes it possible to compute all the rational functions $F_s(t, x)$ for $1 \leq s \leq 7$ in less than a minute.

4. FURTHER REMARKS

While computing with the polynomials $H_m^{(s)}$ we experimentally discovered the following amusing facts.

Fact 1. *For all $s \geq 1$, we have:*

$$\begin{aligned} H_0^{(s)}(x) &= 1, \\ H_1^{(s)}(x) &= x + 1, \\ H_2^{(s)}(x) &= x^2 + L_{2s}x + 1, \end{aligned}$$

where L_n denotes the n -th element of the Lucas sequence defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$.

This is easy to prove.

Fact 2. *$H_m^{(s)}(x)$ has only non-negative coefficients, for all $m \geq 0$ and $s \geq 1$.*

This suggests that the coefficients of the polynomials $H_m^{(s)}(x)$ could admit a nice combinatorial interpretation.

For instance, the coefficient of x^1 in $H_m^{(s)}(x)$ is equal to the trace of the matrix M^{2s} , where $M = (a_{i,j})$ is the $m \times m$ matrix with $a_{i,j} = 1$ for $i + j \leq m + 1$ and $a_{i,j} = 0$ otherwise.

Fact 3. *The rational function $F_s(t, x) = \sum_{m \geq 0} H_m^{(s)}(x)t^m$ writes $N_s(x, t)/D_s(x, t)$, where $N_s(x, t)$ is an irreducible polynomial in $\mathbb{Q}[t, x]$ of degree $2^s - 1$ in x and $B_{s-1} - 1$ in t and $D_s(x, t)$ is an irreducible polynomial in $\mathbb{Q}[t, x]$ of degree 2^s in x and B_{s-1} in t , where B_n is the central binomial coefficient*

$$B_n = \binom{n}{\lfloor n/2 \rfloor}.$$

Here is a sketch of the proof. We will only prove upper bounds, but a finer analysis could lead to the exact degrees.

First, it is enough to show that the degree in x of the polynomial $P_s(t, x)$ defined in the proof of theorem 1 is at most sB_{s-1} . Now, instead of considering P_s , we study the simpler polynomial

$$C_s(t, x) = \prod_{1 \leq i_1, \dots, i_s \leq 2} \left(t - \alpha_{i_1}(x)\alpha_{i_2}(x) \cdots \alpha_{i_s}(x) \right).$$

We will prove the bound sB_{s-1} on the degree of C_s in x . Adapting the proof to the case of P_s is not difficult.

The starting point is that, when x tends to infinity, $\alpha_1(x)$ grows like x , while $\alpha_2(x)$ grows like x^{-1} . Therefore,

$$C_s(t, x) = \prod_{k=0}^s \left(t - \alpha_1^{s-k}(x)\alpha_2^k(x) \right)^{\binom{s}{k}}$$

grows like $\prod_{k=0}^s \left(t - x^{s-k}(x^{-1})^k \right)^{\binom{s}{k}}$. The degree in x of the latter polynomial is equal to

$$\sum_{k=0}^{\lfloor s/2 \rfloor} \binom{s}{k} (s - 2k) = \binom{s}{\lfloor s/2 \rfloor + 1} \cdot (\lfloor s/2 \rfloor + 1) = sB_{s-1}.$$

REFERENCES

- [1] Alin Bostan, Philippe Flajolet, Bruno Salvy, and Éric Schost. Fast computation of special resultants. *J. Symbolic Comput.*, 41(1):1–29, 2006.
- [2] L. Cerlienco, M. Mignotte, and F. Piras. Suites récurrentes linéaires: propriétés algébriques et arithmétiques. *Enseign. Math. (2)*, 33(1-2):67–108, 1987.
- [3] Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.
- [4] Kenneth B. Stolarsky. Discriminants and divisibility for Chebyshev-like polynomials. In *Number theory for the millennium, III (Urbana, IL, 2000)*, pages 243–252. A K Peters, Natick, MA, 2002.
- [5] Khang Tran. Discriminants of Chebyshev-like polynomials and their generating functions. *Proc. Amer. Math. Soc.*, 137(10):3259–3269 (electronic), 2009.
- [6] A. J. van der Poorten. Some facts that should be better known, especially about rational functions. In *Number theory and applications (Banff, AB, 1988)*, volume 265 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 497–528. Kluwer Acad. Publ., Dordrecht, 1989.

APPENDIX: MAPLE CODE

For completeness, we give a self-contained Maple implementation that produces the generating series.

```

recipoly:=proc(p,t) expand(t^degree(p,t)*subs(t=1/t,p)) end:
newt:=proc(p,t,ord)
local pol, dpol;
  pol:=p/lcoeff(p,t);
  pol:=recipoly(pol,t);
  dpol:=recipoly(diff(pol,t),t);
  map(expand,series(dpol/pol,t,ord))
end:
invnewt:=proc(S,t,ord) series(exp(Int((coeff(S,t,0)-S)/t,t)),t,ord) end:

Fs:=proc(s)
local H, N, S, y, den, num;
  H:=(1-t)/((1-t)^2-x*t);
  H:=subs(x=-x,t=-t,H);
  N:=2^s;
  S:=newt(denom(H),t,N+1);
  S:=series(add(resultant(coeff(S,t,i),y-x^s,x)*t^i,i=0..N),t,N+1);
  den:=convert(invnewt(S,t,N+1),polynom); # denominator
  S:=series(H,t,N);
  S:=series(add(resultant(coeff(S,t,i),y-x^s,x)*t^i,i=0..N),t,N);
  num:=convert(series(den*S,t,N),polynom); # numerator
  collect(subs(y=-x,t=-t,num/den),x,factor)
end proc;

```

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