

Discriminants of Polynomials Related to Chebyshev Polynomials: The “Mutt and Jeff” Syndrome

Khang Tran
University of Illinois at Urbana-Champaign

Abstract

The discriminants of certain polynomials related to Chebyshev polynomials factor into the product of two polynomials, one of which has coefficients that are much larger than the other’s. Remarkably, these polynomials of dissimilar size have “almost” the same roots, and their discriminants involve exactly the same prime factors.

1 Introduction

The discriminants of the Chebyshev $T_n(x)$ and $U_n(x)$ polynomials are given by simple and elegant formulas:

$$\text{Disc}_x T_n(x) = 2^{(n-1)^2} n^n \quad (1.1)$$

and

$$\text{Disc}_x U_n(x) = 2^{n^2} (n+1)^{n-2}. \quad (1.2)$$

Is there anything comparable for linear combinations or integral transforms of Chebyshev polynomials? For a special type of linear combination, formula (1.2) was generalized in [3] to

$$\text{Disc}_x (U_n(x) + kU_{n-1}(x)) = 2^{n(n-1)} a_{n-1}(k)$$

where

$$a_{n-1}(k) = (-1)^n \frac{(2n+1)^n k^n}{(n+1)^2 - n^2 k^2} \left(U_n \left(-\frac{a+1+nk^2}{(2n+1)k} \right) + kU_{n-1} \left(-\frac{n+1+nk^2}{(2n+1)k} \right) \right).$$

For various formulas related to this type of linear combination, see [3] and [5]. What can be said about discriminants in z of an integral transform

$$(Sp)(z) = \frac{1}{2} \int_{-z}^z p'(t)(x - t^2) dt$$

where p is some Chebyshev polynomial? Our goal here is to show that the resulting polynomials have discriminants that factor in a remarkable way. More precisely, when $p = U_{2n-1}(z)$ the discriminant factors into two polynomials, one of which has coefficients that are much larger than the coefficients of the other. Moreover, these two polynomials have “almost” the same roots, and their discriminants involve exactly the same prime factors. For example, when $n = 6$, these two polynomials are

$$\begin{aligned} M(x) &= -143 + 2002x - 9152x^2 + 18304x^3 - 16640x^4 + 5632x^5 \\ J(x) &= -2606483707 + 826014609706x - 10410224034496x^2 \\ &\quad + 40393170792832x^3 - 60482893968640x^4 + 30616119778816x^5. \end{aligned}$$

The discriminants of $M(x)$ and $J(x)$ are $2^{64}3^411^313^4$ and $2^{40}3^411^{35}13^{44}$ respectively. The roots of $M(x)$ rounded to 5 digits are

$$\{0.13438, 0.36174, 0.62420, 0.85150, 0.98272\}$$

whereas those of $J(x)$ are

$$\{0.0032902, 0.13452, 0.36181, 0.62428, 0.85163\}.$$

We notice that in this case, the discriminants of these two polynomials have the same (rather small) prime factors. And except for the root 0.98272 of $M(x)$ and 0.0032902 of $J(x)$, the remaining roots are pairwise close. In fact, for n large we can show, after deleting one root from each of $M(x)$ and $J(x)$, that the remaining roots can be paired in such a way that the distance between any two in a given pair is at most $1/2n^2$.

We call the small polynomial $M(x)$ the “Mutt” polynomial and the large polynomial $J(x)$ the “Jeff” polynomial after two American comic strip characters drawn by Bud Fisher. They were an inseparable pair, one of whom (“Mutt”) was very short compared to the other (“Jeff”). These two names are suggested by Kenneth Stolarsky.

Now, what will happen if we take the discriminant in t of $U'_{2n-1}(t)(x - t^2)$ instead of taking the integral? It is not difficult to show that this discriminant is

$$Cx (U'_{2n-1}(\sqrt{x}))^4.$$

where C is a constant depending on n . As we will see, one remarkable property about the polynomial $U'_{2n-1}(\sqrt{x})$ is that its discriminant has the same factors as the discriminants of our polynomials $M(x)$ and $J(x)$ and its roots are almost the same as those of these two polynomials.

The main results are given by theorems 2, 3 and 4 in sections 4, 5 and 6 respectively. In section 2 we provide the reader with a convenient summary of notations and properties of Chebyshev polynomials, discriminants, and resultants. Section 3 of this paper defines the Mutt and the Jeff polynomials. Section 4 and 5 of this paper analyze the discriminants of $J(x)$ and $M(x)$. Section 6 shows that the roots of these two polynomials are pairwise close after deleting one root from each.

2 Discriminant, Resultant and Chebyshev polynomials

The discriminant of a polynomial $P(x)$ of degree n and leading coefficient γ is

$$\text{Disc}_x P(x) = \gamma^{2n-2} \prod_{i < j} (r_i - r_j)^2 \quad (2.1)$$

where r_1, r_2, \dots, r_n are the roots of $P(x)$. The resultant of two polynomials $P(x)$ and $Q(x)$ of degrees n, m and leading coefficients p, q respectively is

$$\begin{aligned} \text{Res}_x(P(x), Q(x)) &= p^m \prod_{P(x_i)=0} Q(x_i) \\ &= q^n \prod_{Q(x_i)=0} P(x_i). \end{aligned}$$

The discriminant of $P(x)$ can be computed in terms of the resultant between this polynomial and its derivative

$$\text{Disc}_x P(x) = (-1)^{n(n-1)/2} \frac{1}{\gamma} \text{Res}(P(x), P'(x)) \quad (2.2)$$

$$= (-1)^{n(n-1)/2} \gamma^{n-2} \prod_{i \leq n} P'(r_i). \quad (2.3)$$

In this paper we compute the discriminants and resultants of polynomials related to Chebyshev polynomials. Here we review some basic properties of the Chebyshev polynomials. The Chebyshev polynomial of the second kind $U_n(x)$ has the derivative

$$U'_n(x) = \frac{(n+1)T_{n+1} - xU_n}{x^2 - 1}. \quad (2.4)$$

The derivative of the Chebyshev polynomial of the first kind $T_n(x)$ is

$$T'_n(x) = nU_{n-1}(x). \quad (2.5)$$

The connections between these polynomials are given by

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)) \quad (2.6)$$

$$= U_n(x) - xU_{n-1}(x) \quad (2.7)$$

$$= xT_{n-1}(x) - (1 - x^2)U_{n-2}(x). \quad (2.8)$$

Mourad Ismail [6] applied the following theorem from Von J. Schur [7] to compute the generalized discriminants of the generalized orthogonal polynomials:

Theorem 1 *Let $p_n(x)$ be a sequence of polynomials satisfying the recurrence relation*

$$p_n(x) = (a_nx + b_n) - c_n p_{n-2}(x)$$

and the initial conditions $p_0(x) = 1$, $p_1(x) = a_1x + b_1$. Assume that $a_1a_nc_n \neq 0$ for $n > 1$. Then

$$\prod_{p_n(x_i)=0} p_{n-1}(x_i) = (-1)^{n(n-1)/2} \prod_{j=1}^n a_j^{n-2j+1} c_j^{j-1}$$

for $n \geq 1$.

Ismail's idea is that if we can construct $A_n(x)$ and $B_n(x)$ so that

$$p'_n(x) = A_n(x)p_{n-1}(x) + B_n(x)p_n(x).$$

Then

$$\text{Disc}_x p_n(x) = \gamma^{n-2} \prod_{j=1}^n a_j^{n-2j+1} c_j^{j-1} \prod_{p_n(x_i)=0} A_n(x_i).$$

In the special case of the Chebyshev polynomial satisfying the recurrence relation $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ and the differential equation

$$U'_n(x) = \frac{2xnU_n - (n+1)U_{n-1}}{x^2 - 1},$$

we obtain

$$\prod_{U_{2n}(x)=0} U_{2n-1}(x) = 1 \quad (2.9)$$

and

$$\text{Disc}_x U_n(x) = 2^{n^2} (n+1)^{n-2}.$$

For the discriminants of various classes of polynomials, see [1, 2, 3, 4, 5].

3 The Mutt and Jeff polynomial pair

In this section we will show that the discriminant of the integral transform of Chebyshev polynomial factors into the square of the product of the Mutt and Jeff polynomials whose formulas will be provided. While the formula for the Mutt polynomial can be given explicitly in terms of the Chebyshev polynomials, we can only describe the Jeff polynomial by its roots. By integration by parts, we have

$$\begin{aligned}
(SU_{2n-1})(z) &= \frac{1}{2} \int_{-z}^z U'_{2n-1}(t)(x - t^2) dt \\
&= (x - z^2)U_{2n-1}(z) + \int_{-z}^z U_{2n-1}(t)tdt \\
&= (x - z^2)U_{2n-1}(z) + \frac{1}{2} \int_{-z}^z U_{2n}(t) + U_{2n-2}(t)dt \\
&= (x - z^2)U_{2n-1}(z) + \left(\frac{T_{2n+1}(z)}{2n+1} + \frac{T_{2n-1}(z)}{2n-1} \right). \tag{3.1}
\end{aligned}$$

This is a polynomial in z with the leading coefficient

$$-2^{2n-1} + \frac{2^{2n}}{2n+1} = -\frac{(2n-1)}{(2n+1)}2^{2n-1}.$$

The discriminant of this polynomial in z is

$$\begin{aligned}
&C_n \text{Res}_z((SU_{2n-1})(z), U'_{2n-1}(z)(x - z^2)) \\
&= C_n \text{Res}_z((SU_{2n-1})(z), x - z^2) \text{Res}_z((SU_{2n-1})(z), U'_{2n-1}(z)) \\
&= C_n \left((2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x}) \right)^2 \left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i > 0}} (SU_{2n-1})(\zeta_i) \right)^2
\end{aligned}$$

where C_n is a rational number depending only on n and can be different in each occurrence. Also the factors of its numerator and denominator can only be 2 or factors of $2n-1$, $2n+1$. From this, we define the Mutt polynomial

$$M(x) = \frac{(2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x})}{x\sqrt{x}}.$$

Also, we can define, within a plus or minus sign, the Jeff polynomial $J(x) \in \mathbb{Z}[X]$ as the polynomial of degree $n - 1$ whose coefficients are relatively prime and for which

$$C_n \left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i > 0}} (SU_{2n-1})(\zeta_i) \right)^2 = A_n J^2(x) \quad (3.2)$$

where A_n is a suitable rational number.

4 The Discriminant of $J(x)$

In this section, we find the factors of the discriminant of $J(x)$. In particular we will prove the theorem:

Theorem 2 *The discriminant of $J(x)$ has the same prime factors as those of the discriminant of $U'_{2n-1}(\sqrt{x})$. Also*

$$\text{Disc}_x U'_{2n-1}(\sqrt{x}) = 3(2n+1)^{n-2}(2n-1)^{n-3}n^{n-3}2^{2n^2-3n-1}.$$

Since we do not have an explicit formula for $J(x)$, we will compute the discriminant as the square of the product of the distances between the roots. We first note that

$$\left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i > 0}} (SU_{2n-1})(\zeta_i) \right)^2$$

is a polynomial in x whose leading coefficient is

$$\begin{aligned} \pm \prod_{U'_{2n-1}(\zeta_i)=0} U_{2n-1}(\zeta_i) &= \pm \frac{\text{Disc}_x U_{2n-1}(x)}{(2n-1)^{2n-1}} \\ &= \pm \frac{2^{(2n-1)^2} (2n)^{2n-3}}{(2n-1)^{2n-1}}. \end{aligned}$$

Thus from (3.2), the factors of the leading coefficient of $J(x)$ can only be 2 or factors of $(2n - 1)$, $(2n + 1)$, n .

We now consider the roots of $J(x)$. According to the formula (3.1), $J(x)$ has $n - 1$ real roots given by the formula

$$-\frac{T_{2n+1}(\zeta_i)}{(2n+1)U_{2n-1}(\zeta_i)} - \frac{T_{2n-1}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2$$

where $U'_{2n-1}(\zeta_i) = 0$ and $\zeta_i > 0$. The derivative formula (4.1) implies that $2nT_{2n}(\zeta_i) = \zeta_i U_{2n-1}(\zeta_i)$. Thus the equation (2.8) gives

$$\begin{aligned} T_{2n+1}(\zeta_i) &= \zeta_i T_{2n}(\zeta_i) - (1 - \zeta_i^2)U_{2n-1}(\zeta_i) \\ &= U_{2n-1}(\zeta_i) \left(\zeta_i^2 \left(1 + \frac{1}{2n}\right) - 1 \right). \end{aligned}$$

Hence the roots of $J(x)$ can be written as below:

$$\begin{aligned} &-\frac{T_{2n+1}(\zeta_i)}{(2n+1)U_{2n-1}(\zeta_i)} - \frac{T_{2n-1}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2 \\ &= \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \frac{T_{2n+1}(\zeta_i)}{U_{2n-1}(\zeta_i)} - \frac{2\zeta_i T_{2n}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2 \\ &= \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \left(\zeta_i^2 \left(1 + \frac{1}{2n}\right) - 1 \right) - \frac{\zeta_i^2}{n(2n-1)} + \zeta_i^2 \\ &= \zeta_i^2 - \frac{2}{(2n+1)(2n-1)}. \end{aligned} \tag{4.1}$$

Thus from the definition of discriminant as the product of differences between roots, it suffices to consider the discriminant of $U'_{2n-1}(\sqrt{x})$. The formulas (2.4) and (2.5) give

$$\begin{aligned} 2\sqrt{x}U''_{2n-1}(\sqrt{x}) &= -\frac{2\sqrt{x}}{x-1}U'_{2n-1}(\sqrt{x}) + \frac{1}{x-1} (4n^2U_{2n-1}(\sqrt{x}) - \sqrt{x}U'_{2n-1}(\sqrt{x}) - U_{2n-1}(\sqrt{x})) \\ &= \frac{-3\sqrt{x}}{x-1}U'_{2n-1}(\sqrt{x}) + \frac{4n^2-1}{x-1}U_{2n-1}(\sqrt{x}). \end{aligned} \tag{4.2}$$

We note that $U'_{2n-1}(\sqrt{x})$ is a polynomial of degree $n - 1$ with the leading coefficient

$$(2n-1)2^{2n-1}.$$

From the definition of discriminant in terms of resultant (2.3) and the formulas (1.2) and (4.2), we have

$$\begin{aligned}
\text{Disc}_x U'_{2n-1}(\sqrt{x}) &= (2n-1)^{n-3} 2^{(2n-1)(n-3)} \prod_{U'_{2n-1}(\sqrt{x_i})=0} \frac{4n^2-1}{x_i-1} \frac{1}{2\sqrt{x}} U_{2n-1}(\sqrt{x}) \\
&= (2n-1)^{n-3} 2^{(2n-1)(n-3)} \prod_{\substack{U'_{2n-1}(x_i)=0 \\ x_i>0}} \frac{4n^2-1}{2x_i(x_i^2-1)} U_{2n-1}(x_i) \\
&= \frac{(4n^2-1)^{n-1}}{2^{n-1} \sqrt{2n/(2n-1)} 2^{2n-1} U'_{2n-1}(1)/(2n-1) 2^{2n-1}} \sqrt{\text{Disc} U_{2n-1}(x)} \\
&\quad \times \frac{(2n-1)^{n-3} 2^{(2n-1)(n-3)}}{\sqrt{(2n-1)^{2n-1} 2^{(2n-1)(2n-2)}}} \\
&= \frac{(4n^2-1)^{n-1}}{2^{n-1} \sqrt{2n/(2n-1)} 2^{2n-1} U'_{2n-1}(1)/(2n-1) 2^{2n-1}} \\
&\quad \times 2^{(2n-1)^2/2} (2n)^{(2n-3)/2} \frac{\sqrt{2n-1}}{(2n-1)^3 2^{2(2n-1)}} \\
&= (2n-1)^{n-2} (2n+1)^{n-1} n^{n-2} 2^{n(2n-3)} / U'_{2n-1}(1),
\end{aligned}$$

where $U'_{2n-1}(1)$ can be computed from its trigonometric definition:

$$\begin{aligned}
U'_{2n-1}(1) &= -\lim_{\theta \rightarrow 0} \frac{2n \cos 2n\theta \sin \theta - \sin 2n\theta \cos \theta}{\sin^3 \theta} \\
&= -\lim_{\theta \rightarrow 0} \frac{2n(1-2n^2\theta^2)(\theta-\theta^3/6) - (2n\theta-8n^3\theta^3/6)(1-\theta^2/2)}{\theta^3} \\
&= -\frac{2}{3}n(4n^2-1).
\end{aligned}$$

Thus

$$\text{Disc}_x U'_{2n-1}(\sqrt{x}) = 3(2n+1)^{n-2} (2n-1)^{n-3} n^{n-3} 2^{2n^2-3n-1}.$$

Thus $\text{Disc}_x J(x)$ has factors 2, 3, and factors of powers of n , $2n-1$, $2n+1$.

5 The discriminant of $M(x)$

In this section, we will present an explicit formula for the discriminant of $M(x)$ and show that this discriminant also has factors 2, 3, and factors of powers of n , $2n-1$, $2n+1$. In particular, we will prove the theorem:

Theorem 3 *The discriminant of $M(x)$ is*

$$\text{Disc}_x M(x) = \pm(2n-1)^{n-3}(2n+1)^{n-2}2^{2n^2-n-5}3n^{n-3}.$$

We note that $M(x)$ is a polynomial in x of degree $n-1$ whose leading coefficient is $(2n-1)2^{2n}$. To compute $M'(x)$ we notice the following fact:

$$(2n-1)T_{2n+1}(x) + (2n+1)T_{2n-1}(x) = 2(2n-1)(2n+1) \int_0^x tU_{2n-1}(t)dt. \quad (5.1)$$

From this, we have

$$M'(x) = -\frac{3}{2} \frac{M(x)}{x} + \frac{(2n+1)(2n-1)}{x\sqrt{x}} U_{2n-1}(\sqrt{x}).$$

Thus the definition of discriminant (2.3) yields

$$\begin{aligned} \text{Disc}_x(M(x)) &= \pm(2n-1)^{n-3}2^{2n(n-3)} \\ &\quad \times (2n-1)^{n-1}(2n+1)^{n-1} \frac{(2n-1)2^{2n}}{M(0)} \prod_{M(x_i)=0} x_i^{-1/2} U_{2n-1}(\sqrt{x_i}). \end{aligned}$$

where the value of the free coefficient of $M(x)$ can be obtained from (5.1)

$$M(0) = \pm 4(2n-1)(2n+1)n/3.$$

By substituting this value to the equation above, we obtain

$$\text{Disc}_x M(x) = \frac{\pm 2^{2n^2-4n-2}(2n-1)^{2n-4}(2n+1)^{n-2}3}{n} \prod_{M(x_i)=0} x_i^{-1/2} U_{2n-1}(\sqrt{x_i}).$$

To compute the product above, we follow an idea from Jemal Gishe and Mourad Ismail [5] by writing $M(x)$ as a linear combination of $U_{2n-1}(x)$ and $U_{2n}(x)$. In particular this combination can be obtained from (2.6) and (2.7) as below:

$$\begin{aligned} (2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x}) &= -2T_{2n+1}(\sqrt{x}) + 2\sqrt{x}(2n+1)T_{2n}(\sqrt{x}) \\ &= -U_{2n+1}(\sqrt{x}) + U_{2n-1}(\sqrt{x}) \\ &\quad + 2\sqrt{x}(2n+1)(U_{2n}(\sqrt{x}) - \sqrt{x}U_{2n-1}(\sqrt{x})) \\ &= 4n\sqrt{x}U_{2n}(\sqrt{x}) - (2x(2n+1) - 2)U_{2n-1}(\sqrt{x}). \end{aligned}$$

Thus

$$\begin{aligned}
\prod_{M(x_i)=0} x^{-1/2} U_{2n-1}(\sqrt{x}) &= \frac{2^{(2n-1)(n-1)}}{(2n-1)^{n-1} 2^{2n(n-1)}} \prod_{x_i^{-1/2} U_{2n-1}(\sqrt{x_i})=0} M(x_i) \\
&= \frac{2^{(2n-1)(n-1)}}{(2n-1)^{n-1} 2^{2n(n-1)}} 4^{n-1} n^{n-1} \\
&\quad \times \frac{2^{2n-1}}{2n} \prod_{x^{-1/2} U_{2n-1}(\sqrt{x})=0} U_{2n}(\sqrt{x}) \\
&= \frac{n^{n-2} 2^{3n-3}}{(2n-1)^{n-1}} \prod_{\substack{U_{2n-1}(x_i)=0 \\ x_i > 0}} U_{2n}(x_i) \\
&= \pm \frac{n^{n-2} 2^{3n-3}}{(2n-1)^{n-1}}
\end{aligned}$$

where the last equality is obtained from (2.9). Hence

$$\text{Disc}_x M(x) = \pm (2n-1)^{n-3} (2n+1)^{n-2} 2^{2n^2-n-5} 3n^{n-3}.$$

From this we conclude that the discriminants of $M(x)$ and $J(x)$ have the same factors 2, 3, and all the factors of n , $2n-1$, $2n+1$.

6 The roots of $M(x)$ and $J(x)$

In this section we will show that the roots of $M(x)$ and $J(x)$ are pairwise similar except for one pair. In particular, we will prove the theorem:

Theorem 4 *Suppose n is sufficiently large. Then for every root x_0 of $J(x)$ except the smallest root, there is a root of $M(x)$ in the interval*

$$[x_0 - 3/(10n^2), x_0 + 1/(2n^2)].$$

We first show that with one exception, the roots of $M(x)$ and $U'_{2n-1}(\sqrt{x})$ are pairwise close. From the definition, the roots of these two polynomial are positive real numbers. To simplify the computations, we consider $U'_{2n-1}(x)$ and the polynomial

$$R(x) = (2n-1)T_{2n+1}(x) + (2n+1)T_{2n-1}(x)$$

whose roots (except 0) are the square roots of the positive roots of $J(x)$ and $M(x)$ respectively. And it will suffice to consider the positive roots of $R(x)$ and $U'_{2n-1}(x)$.

Let ζ be a positive root of $U'_{2n-1}(x)$. We will show that for a certain small value $\delta > 0$, the quantities $R(\zeta)$ and $R(\zeta - \delta)$ have different signs and thus $R(x)$ admits a root in the small interval $(\zeta - \delta, \zeta)$. First, we observe that the equation (4.1) gives

$$R(\zeta) = 2U_{2n-1}(\zeta).$$

For some $t \in (\zeta - \delta, \zeta)$, we have

$$R(\zeta - \delta) = R(\zeta) - R'(t)\delta.$$

where

$$R'(t) = 2(2n+1)(2n-1)tU_{2n-1}(t).$$

Thus

$$R(\zeta - \delta) = 2U_{2n-1}(\zeta) - 2(2n+1)(2n-1)tU_{2n-1}(t)\delta. \quad (6.1)$$

To prove $R(\zeta - \delta)$ and $R(\zeta) = 2U_{2n-1}(\zeta)$ have different signs, it remains to show that $2(2n+1)(2n-1)tU_{2n-1}(t)\delta$ is sufficiently large in magnitude and has the same sign as $R(\zeta)$. To ensure this fact, we first impose the following two conditions on δ :

$$\zeta\delta > \frac{A}{(2n+1)(2n-1)} \quad (6.2)$$

$$\zeta > 6\delta \quad (6.3)$$

where the value of A and the existence of δ with respect to these conditions will be determined later. With these conditions, we obtain the following lower bound for $R'(t)\delta$:

$$\begin{aligned} |2(2n+1)(2n-1)U_{2n-1}(t)t\delta| &> 2(2n+1)(2n-1)|U_{2n-1}(t)|(\zeta - \delta)\delta \\ &> \frac{5}{3}(2n+1)(2n-1)|U_{2n-1}(t)|\zeta\delta \\ &> \frac{5A}{3}|U_{2n-1}(t)|. \end{aligned} \quad (6.4)$$

We now need to show that $U_{2n-1}(t)$ is not too small compared to $U_{2n-1}(\zeta)$. The Taylor formula gives

$$U_{2n-1}(t) = U_{2n-1}(\zeta) + U''_{2n-1}(\zeta - \epsilon)(t - \zeta)^2 \quad (6.5)$$

where $0 < \epsilon < \zeta - t$. From (2.4), we have, for $-1 < x < 1$, the following trivial bound for $U'_n(x)$:

$$|U'_n(x)| < \frac{2n}{1-x^2}. \quad (6.6)$$

Thus the equation (4.2), with \sqrt{x} replaced by x , implies that for large n

$$\begin{aligned} |U''_{2n-1}(\zeta - \epsilon)| &\sim \frac{4n^2}{1 - (\zeta - \epsilon)^2} |U_{2n-1}(\zeta - \epsilon)| \\ &< \frac{4n^2}{1 - \zeta^2} |U_{2n-1}(\zeta)|. \end{aligned} \quad (6.7)$$

We impose another condition on δ :

$$\delta < \frac{\sqrt{1 - \zeta^2}}{4n}. \quad (6.8)$$

With this condition, we have

$$|U''_{2n-1}(\zeta - \epsilon)(t - \zeta)^2| < \frac{1}{4} |U_{2n-1}(\zeta)|.$$

Now the equation (6.5) implies that $U_{2n-1}(t)$ and $U_{2n-1}(\zeta)$ have the same sign and

$$|U_{2n-1}(t)| > \frac{3}{4} |U_{2n-1}(\zeta)|.$$

This inequality combined with (6.1) and (6.4) show that $R(\zeta - \delta)$ and $R(\zeta)$ have different signs if

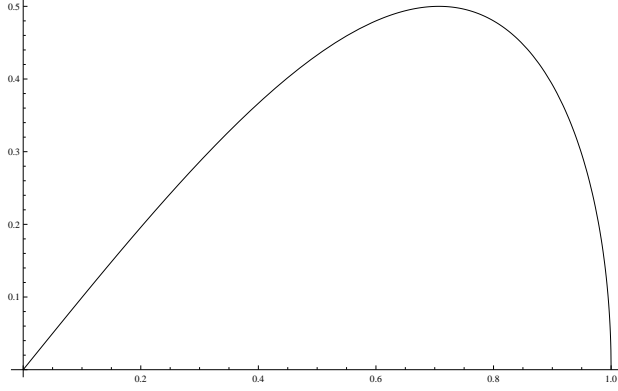
$$A > \frac{8}{5}.$$

Thus $R(x)$ has a root in the interval $(\zeta - \delta, \zeta)$.

Next we will show that δ exists with respect to all the imposed conditions and that its value is small as soon as ζ is not the smallest positive root of $U'_{2n-1}(x)$. Let ζ be such a root. From (6.2), (6.3) and (6.8), it suffices to show ζ satisfies the identity

$$\frac{A}{(2n-1)(2n+1)} < \min \left(\frac{1}{4n} \zeta \sqrt{1 - \zeta^2}, \frac{\zeta^2}{6} \right). \quad (6.9)$$

The graph of the function $\zeta \sqrt{1 - \zeta^2}$ is given below:



From this graph, it suffices to show that ζ is not too small and not too close to 1. Let $\zeta_1 > 0$ be the second smallest root of $U'_{2n-1}(x)$. This root has an upper bound

$$\zeta_1 < \cos \frac{\pi(n-2)}{2n} \sim \frac{\pi}{n}$$

since the right side is the second smallest positive root of $U_{2n-1}(x)$. The smallest positive root of $U_{2n-1}(x)$ is

$$\cos \frac{\pi(n-1)}{2n} \sim \frac{\pi}{2n}.$$

Next the trivial lower bound $\zeta_1 > \pi/2n$ does not guaranty the existence of A in (6.9). We need a slightly better lower bound.

For some $\xi \in (\pi/2n, \zeta_1)$ we have

$$\left(\zeta_1 - \frac{\pi}{2n}\right)U'_{2n-1}(\xi) = U_{2n-1}(\zeta_1).$$

The bound (6.6) gives

$$\begin{aligned} \zeta_1 - \frac{\pi}{2n} &> \frac{|U_{2n-1}(\zeta_1)|(1-\xi^2)}{2(2n-1)} \\ &> \frac{1 - (\pi/n)^2}{2(2n-1)} \\ &\sim \frac{1}{4n}, \end{aligned}$$

so

$$\zeta_1 > \frac{2\pi + 1}{4n}.$$

We now see that for n large ζ_1 satisfies the condition (6.9) if

$$A < \frac{2\pi + 1}{4}.$$

Similarly, let ζ_2 be the largest root of $U'_{2n-1}(x)$. The trivial upper bound $\zeta_2 < \cos \pi/2n$, the largest root of $U_{2n-1}(x)$, does not guaranty the existence of A . However for some $\xi \in (\zeta_2, \cos \pi/2n)$ we have

$$\left(\cos \frac{\pi}{2n} - \zeta_2\right)^2 U''_{2n-1}(\xi) = U_{2n-1}(\zeta_2).$$

The approximation (6.7) yields

$$\begin{aligned} \left(\cos \frac{\pi}{2n} - \zeta_2\right)^2 &\sim \frac{|U_{2n-1}(\zeta_2)|(1 - \xi^2)}{4n^2 |U_{2n-1}(\xi)|} \\ &> \frac{(1 - \cos^2 \pi/2n)}{4n^2} \\ &\sim \frac{\pi^2}{16n^4}. \end{aligned}$$

Thus

$$\begin{aligned} \sqrt{1 - \zeta_2^2} &> \sqrt{1 - \left(\cos \frac{\pi}{2n} - \frac{\pi}{4n^2}\right)^2} \\ &\sim \sqrt{\frac{\pi^2}{4n^2} + \frac{\pi}{2n^2}} \\ &> \frac{2\pi + 1}{4n}. \end{aligned}$$

Now the condition(6.9) is satisfied provided that $A < (2\pi + 1)/4$.

Thus we can let A be any value in the interval $(8/5, (2\pi + 1)/4)$. Since for large n , A and δ are arbitrarily bigger than $8/5$ and $A/4n^2\zeta$ respectively, we have that for every root ζ of $U'_{2n-1}(x)$ except the smallest root, there is a root of $R(x)$ in the small interval

$$\left[\zeta - \frac{8}{20n^2\zeta}, \zeta\right).$$

This implies that in each pair, the root ζ^2 of $U'_{2n-1}(\sqrt{x})$ is bigger than the root of x_i of $M(x)$ by at most

$$\frac{8}{20n^2\zeta}(\zeta + \sqrt{x_i}) < \frac{8}{10n^2}.$$

Combining this fact with (4.1), we conclude that for every root x_0 of $J(x)$ except the smallest root, there is a root of $M(x)$ in the interval

$$\left[x_0 - \frac{3}{10n^2}, x_0 + \frac{1}{2n^2}\right)$$

for large n . This concludes the proof.

Remark: Similar Mutt and Jeff phenomena seem to occur for the integral transform of Legendre polynomials.

References

- [1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [2] T. M. Apostol, The resultants of the cyclotomic polynomials $F_m(ax)$ and $F_n(bx)$, *Math. Comp.* 29 (1975), 1-6.
- [3] K. Dilcher and K. B. Stolarsky, Resultants and Discriminants of Chebyshev and related polynomials, *Transactions of the Amer. Math. Soc.* 357 (2004), 965-981.
- [4] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhuser Boston, Boston, 1994.
- [5] J. Gishe and M. E. H. Ismail, Resultants of Chebyshev Polynomials, *Journal for Analysis and its Applications* 27 (2008), no. 4, 499-508.
- [6] M. E. H. Ismail, Difference equations and quantized discriminants for q -orthogonal polynomials. *Advances in Applied Mathematics* 30 (2003), 562-589.
- [7] J. Schur, Affektlose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynomes, *J. Reine Angew. Math.* 165 (1931), 52-58.