Enumerating Rook and Queen Paths

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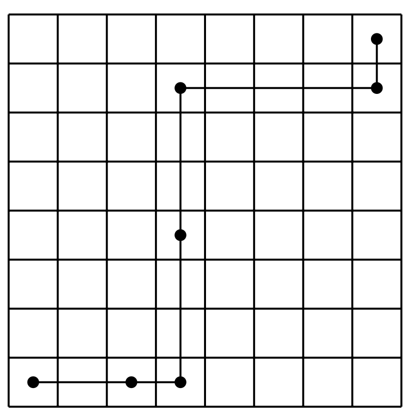
Abstract

How many ways can a chess Rook or Queen move from a corner cell to the opposite corner cell of an arbitrary size, arbitrary dimensional chessboard, assuming that the piece moves closer to the goal cell at each step? Recurrence relations, generating functions, and asymptotic formulas have already been given for Rook paths in dimension 2. We revisit these and determine similar results for Queen paths. We also describe some results and open questions concerning the number of Rook and Queen paths in higher dimensions. As a consequence of our analysis, we find an asymptotic formula for the number of Nim games that start with an arbitrary number of equal-size piles of stones.

A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an 8×8 chessboard? Assume that the Rook moves right or up at each step. An example of a Rook path is shown below.



More generally, we can count lattice paths from $(0,0)$ to (n,n) with steps of the form $(x,0)$ or $(0,y)$, where x and y are positive integers. Some similar problems are well known. For example, central Delannoy numbers count the number of King paths, i.e., where the steps are of the form $(1,0)$, $(0,1)$, or $(1,1)$; Catalan numbers count paths that move in steps of $(1,0)$ or $(0,1)$ and never go above the line $y = x$; and Schröder numbers count King paths that do not go above the line $y = x$. [**8**]

The Rook path problem is simply solved by generalizing to find the number of paths from $(0,0)$ to any given square on an arbitrary size board, that is, to any point (m,n). Let $a(m,n)$ be the number of such paths, where $m, n \geq 0$. We decree that $a(0,0) = 1$. Of course, $a(m,n) = a(m,n)$. For m or n positive, $a(m,n)$ is equal to the sum of the horizontal and vertical predecessors of (m,n), since the Rook arrives at (m,n) from one of the squares to its left or below it. For example, $a(2,1) = 1 + 2 + 2 = 5$. From the table below, we see that the number of Rook paths from the lower-left corner to the upper-right corner of the 8×8 chessboard is $a(7,7) = 470010$.

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
64	320	1328	4864	16428	52356	159645	470010	...
32	144	560	1944	6266	19149	56190	159645	...
16	64	232	760	2329	6802	19149	52356	...
8	28	94	289	838	2329	6266	16428	...
4	12	37	106	289	760	1944	4864	...
2	5	14	37	94	232	560	1328	...
1	2	5	12	28	64	144	320	...
1	1	2	4	8	16	32	64	...

Our recurrence formula for $a(m,n)$ requires a variable number of preceding terms. In contrast, we also notice a recurrence relation requiring only three preceding terms:

$$a(0,0) = 1,\ a(0,1) = 1,\ a(1,0) = 1,\ a(1,1) = 2;$$
$$a(m,n) = 2a(m-1,n) + 2a(m,n-1) - 3a(m-1,n-1), \quad m \geq 2 \text{ or } n \geq 2.$$

(Assume that $a(m,n) = 0$ for m or n negative.) This recurrence is easy to prove by inclusion–exclusion but it will also follow from the forthcoming theorem.

For ease of reading, we indicate the recurrence relation with an array of its coefficients.

$$\begin{array}{cc} -2 & 1 \\ 3 & -2 \end{array}$$

The recurrence formula yields a rational ordinary generating function for the doubly-infinite sequence $\{a(m,n)\}$, namely,

$$\sum_{m \geq 0,\, n \geq 0} a(m,n) s^m t^n = \frac{1 - s - t + st}{1 - 2s - 2t + 3st}.$$

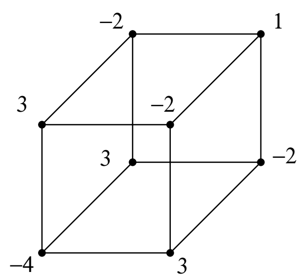
The denominator is implied by the recurrence relation. The numerator is obtained by multiplying the denominator by the polynomial that represents the initial values, $1 + s + t + 2st$, and keeping only those monomials with exponents of s and t both less than 2.

The generating function for 2-D Rook paths yields the following direct computational formula, which is impractical for m and n large:

$$a(m, n) = \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{p+q}{p} \binom{m-1}{p-1} \binom{n-1}{q-1}.$$

We will show a better way to calculate $a(m, n)$ a little later.

We can generalize the Rook path problem to 3-dimensional space. How many ways can a Rook move from $(0, 0, 0)$ to (m, n, o), where each step is a positive integer multiple of $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$? The coefficients for a linear recurrence relation are indicated below.



The corresponding generating function is

$$\frac{(1-s)(1-t)(1-u)}{1 - 2(s + t + u) + 3(st + su + tu) - 4stu}.$$

To obtain the numerator, we multiply the denominator by the polynomial representing the initial values, $1 + s + t + u + 2st + 2su + 2tu + 6stu$, and keep only those monomial terms with no variable raised to a power higher than 1.

We could do a similar analysis of higher-dimensional Rook paths, but the result will follow easily from the forthcoming theorem. We just note that in any dimension the recurrence is depth one wherein each variable is decremented by 0 or 1, and each coefficient is equal to $(n+1)(-1)^n$, where n is the number of variables decremented. The initial values are

$$a(c_1, c_2, \ldots, c_d) = j!,$$

where each c_i equals 0 or 1, and j is the number of i such that $c_i = 1$.

The pattern of the rational generating function for Rook paths applies to a general type of lattice path enumeration problem. Consider a 1-dimensional version with two "basic steps." Suppose that there are two stamp rolls, one with 1-cent stamps and the other with 2-cent stamps. Let $a(n)$ be the number of ways to make postage of n cents by taking strips of stamps from the two rolls. The order of the strips and the number of stamps per strip matter. For example, $a(4) = 15$ since

$$\begin{aligned}4 &= (1) + (1) + (1) + (1) = (1 + 1) + (1) + (1) = (1) + (1 + 1) + (1)\\ &= (1) + (1) + (1 + 1) = (1 + 1) + (1 + 1) = (1 + 1 + 1) + (1)\\ &= (1) + (1 + 1 + 1) = (1 + 1 + 1 + 1) = (2) + (1) + (1)\\ &= (1) + (2) + (1) = (1) + (1) + (2) = (2) + (1 + 1)\\ &= (1 + 1) + (2) = (2) + (2) = (2 + 2).\end{aligned}$$

These expressions are compositions of 4 formed with 1's and 2's in which runs of like numbers are grouped arbitrarily. It's easy to show by inclusion–exclusion that the ordinary generating function for $\{a(n)\}$ is

$$\frac{(1 - x^1)(1 - x^2)}{1 - 2(x^1 + x^2) + 3(x^1 \cdot x^2)}.$$

The central problem is to count lattice paths in d dimensions from the origin to a point $(p_1, \ldots, p_d)$, such that each step is a positive integer multiple of a basic step of the form $u_i = (u_{i1}, \ldots, u_{id})$, where $1 \leq i \leq k$.

As an example of the method, to obtain the generating function for 2-D Rook paths, we start with the generating function $1/(1 - x - y)$, which counts sequences of length n having some number of x's and a complementary number of y's (the total number of x's and y's is n). For Rook paths, we allow an arbitrary step length in each direction. This amounts to replacing x by $x/(1 - x)$ and y by $y/(1 - y)$. Hence the desired generating function is

$$\frac{1}{1 - (x/(1 - x)) - (y/(1 - y))}.$$

As another example, to obtain the generating function for 2-D Queen paths (which we will define shortly), we start with the generating function $1/(1 - x - y - z)$ and replace x by $x/(1 - x)$, y by $y/(1 - y)$, and z by $xy/(1 - xy)$.

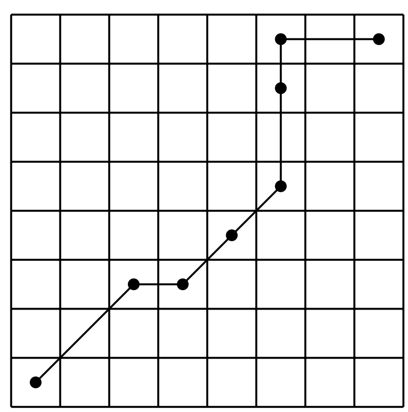
The general situation works just as in these examples. The following theorem gives the rational generating function (which we have rewritten using elementary symmetric polynomials).

THEOREM. For $d \geq 1$ and $1 \leq i \leq k$, let $u_i = (u_{i1}, \ldots, u_{id})$ be a nonzero d-tuple of nonnegative integers. Then the number of lattice paths in d dimensions that go from $(0, \ldots, 0)$ to $(p_1, \ldots, p_d)$, where the p_i are nonnegative integers, and each step is a positive integer multiple of one of the u_i, is the coefficient of x^p given by the rational generating function

$$\frac{\prod_{i=1}^{k}(1 - x^{u_i})}{\sum_{j=0}^{k}(-1)^j (j+1)\sigma_j},$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ and σ_j is the jth elementary symmetric polynomial in the indeterminates x^{u_i}.

A chess Queen can move any number of squares horizontally, vertically, or diagonally in one step. How many ways can a Queen move from the lower-left corner to the upper-right corner of an 8×8 chessboard, assuming that the Queen moves up, right, or diagonally up-right at each step? A Queen path is shown below.



Let $b(m, n)$ be the number of Queen paths from $(0, 0)$ to (m, n), such that at each step the Queen moves up, right, or up-right. As with the Rook paths, we make a table of the number of paths to each square. We calculate each entry in the table by adding all the entries to the left of, below, and diagonally left-below the entry, since the Queen must arrive from one of the aforementioned squares. For example, $b(2, 2) = 2 + 7 + 2 + 7 + 1 + 3 = 22$. The number of paths from one corner to the opposite corner of a chessboard is $b(7, 7) = 1499858$.

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
64	464	2392	10305	39625	140658	470233	1499858	…
32	208	990	3985	14430	48519	154352	470233	…
16	92	401	1498	5079	16098	48519	140658	…
8	40	158	543	1712	5079	14430	39625	…
4	17	60	188	543	1498	3985	10305	…
2	7	22	60	158	401	990	2392	…
1	3	7	17	40	92	208	464	…
1	1	2	4	8	16	32	64	…

For Queen paths, the basic steps are given by s, t, and st. Thus, by our theorem, the generating function is

$$\frac{(1-s)(1-t)(1-st)}{1-2(s+t+st)+3(s\cdot t+s\cdot st+t\cdot st)-4(s\cdot t\cdot st)}.$$

From the denominator, we get a linear recurrence relation with constant coefficients for the sequence $\{b(m,n)\}$.

$$\begin{array}{rrr} 0 & -2 & 1 \\ 3 & 1 & -2 \\ -4 & 3 & 0 \end{array}$$

Written out fully, the recurrence formula for the number of Queen paths is

$$\begin{aligned}
&b(0,0)=1,\ b(0,1)=1,\ b(0,2)=2,\\
&b(1,0)=1,\ b(1,1)=3,\ b(1,2)=7,\\
&b(2,0)=2,\ b(2,1)=7,\ b(2,2)=22;\\
&b(m,n)=2b(m-1,n)+2b(m,n-1)-b(m-1,n-1)-3b(m-2,n-1)\\
&-3b(m-1,n-2)+4b(m-2,n-2),\quad m\geq 2 \text{ or } n\geq 2.
\end{aligned}$$

(Assume that $b(m,n)=0$ for m or n negative.)

In any dimension, a Queen path is at each step a positive integer multiple of a vector that consists entirely of 0's and 1's (but not all 0's). The Queen in d dimensions is similar to the Rook in 2^d-1 dimensions with generators the power set of d variables (except the empty set). The depth of the recurrence relation for the Queen in d dimensions is 2^{d-1}. For example, for 3-D Queen paths, the pattern is the same as for Rook paths in 7 dimensions with generators x, y, z, xy, xz, yz, xyz, and the depth of the recursion is 4. As an exercise, you may wish to determine the pattern of (constant) coefficients of the depth four linear recurrence relation for the 3-D Queen.

The (main) diagonal sequence (where all coordinates are equal) and pure sequences (where all coordinates but one are fixed) of a multivariate sequence with a rational generating function are D-finite, i.e., each has a generating function which satisfies a linear equation with a finite number of derivatives and polynomial coefficients; equivalently, each satisfies a linear homogeneous recurrence relation with polynomial coefficients.[**4**] In the case of Rook and Queen paths, we can find the diagonal generating functions, recurrence relations, and asymptotics for dimension two but we don't know very much for higher dimensions.

Let $a_n = a(n,n)$ be the nth diagonal element of the sequence for the 2-D Rook. The sequence $\{a_n\}$ is A051708 in the Encyclopedia of Integer Sequences (EIS):

$$1, 2, 14, 106, 838, 6802, 56190, 470010, \dots.$$

(In the database, the first term is $a(1)$, whereas ours is a_0.) The following generating function for this sequence was conjectured by Ralf Stephan, who did a computer survey of many sequences, performing elementary operations and looking for matches in the database:

$$f(x) = \frac{1}{2}\left(1 + \frac{(1-x)}{\sqrt{(1-x)(1-9x)}}\right).$$

The method for confirming this generating function for the diagonal sequence is to make the change of variables $t = x/s$ (so that $st = x$). Then the diagonal generating function is the coefficient of s^0, because no s occurs in it. We can find this coefficient by using partial fractions and Laurent series (the method is explained in [**9**]).

Let

$$g(x) = 2f(x) - 1 = \frac{\sqrt{1-x}}{\sqrt{1-9x}}.$$

Note that f and g generate sequences satisfying the same recurrence relation but with different initial values. By logarithmic differentiation,

$$g'(x)(1-x)(1-9x) = 4g(x),$$

and we can read off a recurrence formula for $\{a_n\}$ directly:

$$\begin{aligned}
&a_0 = 1,\ a_1 = 2;\\
&a_n = ((10n-6)a_{n-1} - (9n-18)a_{n-2})/n, \quad n \geq 2.
\end{aligned}$$

This recurrence relation was found by Curtis Coker [2]. It is similar to the one for Delannoy numbers, which count King paths:

$$\begin{aligned}
&d_0 = 1,\ d_1 = 3;\\
&d_n = ((6n-3)d_{n-1} - (n-1)d_{n-2})/n, \quad n \geq 2.
\end{aligned}$$

Open Question 1: Is there a combinatorial proof of the recurrence formula for the a_n? Paul Peart and Wen-Jin Woan found a combinatorial proof of the recurrence formula for the Delannoy numbers **[5]**.

How fast do the numbers a_n grow? To answer this question, we examine $\sqrt{1-x}/\sqrt{1-9x}$ at the singularity $x = 1/9$. The coefficient of x^n in the expansion of $(1-9x)^{-1/2}$ is (using Stirling's approximating for $n!$)

$$\binom{-1/2}{n}(-1)^n 9^n \to 9^n/\sqrt{\pi n}.$$

Using "transfer" (see, e.g., **[3]**), we obtain

$$a_n \sim c\, 9^n/(2\sqrt{\pi n}),$$

where $c = \sqrt{1-1/9} = 2\sqrt{2}/3$.

We can also derive a recurrence relation for pure sequences (in which one of the coordinates is constant). From the generating function

$$f(x,y) = \frac{(1-x)(1-y)}{1-2(x+y)+3xy},$$

we find (by induction) that

$$\frac{\partial^m f}{\partial x^m} = \frac{m!(-1)^{m+1}(y-1)^2(3y-2)^{m-1}}{(1-2(x+y)+3xy)^{m+1}},$$

and hence the mth coefficient of the generating function for the pure sequence, call it a_m, is given by

$$a_m = \frac{1}{m!}\frac{\partial^m f}{\partial x^m}\bigg|_{x=0} = \frac{(y-1)^2(3y-2)^{m-1}}{(2y-1)^{m+1}}.$$

Define $f(y)$ to be the function on the right. Then

$$f'(y) = \frac{(y-1)(-3+n(y-1)+5y)}{(6y^2-7y+2)^2}\frac{(3y-2)^m}{(2y-1)^m}.$$

It follows that

$$f'(y)(6y^3-13y^2+9y-2) = f(y)(-3-m+5y+my),$$

and we can read off a pure recurrence relation:

$$\begin{aligned} 0 =\, & a(m,n)(2n) \\ & + a(m,n-1)(-m-9n+6) \\ & + a(m,n-2)(m+13n-21) \\ & + a(m,n-3)(-6n+18), \quad m \geq 0, n \geq 1. \end{aligned}$$

Since we know the initial values $a(m,0) = 2^{m-1}$, we can compute $a(m,n)$ in $\min(m,n)$ steps, retaining only three values of the sequence in memory at any time.

For 3-D Rook paths, we conjecture (from computer calculations) that the diagonal sequence satisfies the recurrence relation

$$\begin{aligned}
&a_0 = 1,\ a_1 = 6,\ a_2 = 222,\ a_3 = 9918;\\
&a_n = ((121n^3 - 212n^2 + 85n + 6)a_{n-1}\\
&+ (475n^3 - 3462n^2 + 7853n - 5658)a_{n-2}\\
&+ (-1746n^3 + 14580n^2 - 40662n + 37908)a_{n-3}\\
&+ (1152n^3 - 12672n^2 + 46080n - 55296)a_{n-4})/(2n^3 - 2n^2), \quad n \geq 4.
\end{aligned}$$

Open Question 2: What is the linear recurrence relation with polynomial coefficients for the Rook paths diagonal sequence for dimension greater than 2? Can we at least say what its order and degree of polynomial coefficients are?

For the 2-D Queen, the diagonal sequence $\{b_n = b(n,n)\}$ is the EIS sequence A132595:

$$1, 3, 22, 188, 1712, 16098, 154352, 1499858, 14717692, 145509218, \ldots.$$

Let $x = st$. Then the generating function becomes

$$f(x) = \frac{(x-1)(s^2 - (-x-1)s + x)}{(3x-2)s^2 + (-4x^2 + x + 1)s + (3x^2 - 2x)}.$$

Using partial fractions and Laurent series, we obtain

$$f(x) = \frac{(x-1)}{(3x-2)}\left[1 + \frac{1-x}{\sqrt{1-12x+16x^2}}\right].$$

Solving for $1/\sqrt{1-12x+16x^2}$ and taking a derivative yields

$$f(x)(46x^2 - 47x + 11) + f'(x)(48x^4 - 116x^3 + 95x^2 - 29x + 2) = 10x^2 - 15x + 5,$$

and we can read off the recurrence formula:

$$\begin{aligned}
&b_0 = 1,\ b_1 = 3,\ b_2 = 22,\ b_3 = 188;\\
&b_n = ((29n - 18)b_{n-1} + (-95n + 143)b_{n-2}\\
&+ (116n - 302)b_{n-3} + (-48n + 192)b_{n-4})/(2n), \quad n \geq 4.
\end{aligned}$$

By "transfer," we have

$$b_n \sim c(1/r_1)^n/\sqrt{\pi n},$$

where $c = \sqrt{10(3\sqrt{5}-5)}/8$.

Open Question 3: What is the pure linear recurrence relation with polynomial coefficients for 2-D Queen paths?

Open Question 4: What is the linear recurrence relation with polynomial coefficients for the Queen paths diagonal sequence for dimension greater than 2?

To compute the asymptotics of the main diagonal sequence for Rook paths, we apply a special case of a theorem from Robin Pemantle and Mark C. Wilson [6]. Let $\mathbf{x} \in \mathbb{C}^d$ and $\mathbf{n} \in \mathbb{Z}^d$. Suppose that

$$F(\mathbf{x}) = \sum a_{\mathbf{n}} \mathbf{x}^n = \frac{I(\mathbf{x})}{J(\mathbf{x})},$$

where J is holomorphic on an open domain D containing the closure of the domain of convergence of F. A critical point of F for the main diagonal vector is a solution of

$$\begin{aligned} J(\mathbf{x}) &= 0 \\ x_i \partial_j J(\mathbf{x}) &= x_d \partial_d J(\mathbf{x}). \end{aligned}$$

A contributing point of F is a critical point that influences the asymptotics of the main diagonal sequence.

The following proposition, observed by Alexander Raichev and Mark C. Wilson [7], is a consequence of the Pemantle and Wilson Theorem result.

Proposition. Suppose that $J(\mathbf{x})$ is symmetric in $\mathbf{x}$ and $\partial_d J(\mathbf{c}) \neq 0$, where $\mathbf{c} = (c, \ldots, c)$ is the unique contribution point in the positive orthant $(\mathbb{R}^+)^d$. Then

$$a(n, \ldots, n) \sim c^{-nd} b n^{(1-d)/2},$$

with

$$b = \frac{I(\mathbf{c})}{-c\partial_d J(\mathbf{c})\sqrt{(2\pi)^{d-1} d a^{d-1}}},$$

where

$$a = 1 + \frac{c}{\partial_d J}(\partial_d^2 J - \partial_1 \partial_d J)|_{\mathbf{x}=\mathbf{c}}.$$

We now apply the proposition to our d-dimensional Rook paths with

$$\sum a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} = \frac{\prod_{i=1}^{d}(1-x_i)}{\prod_{i=1}^{d}(1-x_i) - \sum_{i=1}^{d} x_i \prod_{j \neq i}(1-x_i)}.$$

The equation $J(c, \ldots, c) = 0$ gives

$$(1-c)^d - dc(1-c)^{d-1} = 0.$$

Thus $c = 1/(d+1)$ or $c = 1$. Since $(1, \ldots, 1)$ is not a convergence point of the power series, the unique contribution point in the positive orthant $(\mathbb{R}^+)^d$ is

$$\left(\frac{1}{d+1}, \ldots, \frac{1}{d+1}\right).$$

The formula in the proposition gives an asymptotic formula for the number of Rook paths from the origin to a diagonal point:

$$a(n, \ldots, n) \sim (d+1)^{dn-1} d^{(d+2)/2} (2\pi n(d+2))^{(1-d)/2}.$$

Rook paths are equivalent to Nim games while Queen paths are equivalent to Wythoff's Nim games. Recall that in Nim, the players alternately remove any number of stones from one of a number of piles. The game ends when the last stone is removed. In Wythoff's Nim, the players in each turn remove the same number of stones from any of the piles. The following observations are not new (at least in the case of Wythoff's Nim; see, e.g., [**1**]):

- Rook paths from $(0, 0, \ldots, 0)$ to $(a_1, a_2, \ldots, a_d)$ are equivalent to Nim games that start with d piles of stones of sizes a_1, a_2, …, a_d.
- Queen paths from $(0, 0, \ldots, 0)$ to $(a_1, a_2, \ldots, a_d)$ are equivalent to Wythoff's Nim games that start with d piles of stones of sizes a_1, a_2, …, a_d.

According to our analysis, we can say that the number of Nim games that start with two piles of n stones satisfies a linear recurrence relation of order two with linear polynomial coefficients. Empirically, the number of Nim games that start with three piles of n stones satisfies a linear recurrence relation of order four with polynomial coefficients of degree 3. The number of Nim games that start with d piles of n stones is given by the asymptotic formula above. The number of Wythoff's Nim games that start with two piles of n stones satisfies a linear recurrence relation of order four with linear polynomial coefficients. There is no consideration of strategy in these results; we are counting all possible games.

Obtaining an asymptotic formula for the number of Queen paths from the origin to a diagonal point via the same type of analysis appears to be quite challenging, since finding the critical point in the positive orthant requires solving the equation

$$1-\binom{d}{1}\frac{x}{1-x}-\binom{d}{2}\frac{x^2}{1-x^2}-\cdots-\binom{d}{d}\frac{x^d}{1-x^d}=0,$$

and no straightforward solution is apparent.

Open Question 5: For Queen paths in d dimensions, what is the growth rate of the diagonal sequence?

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