

The root distribution of polynomials with a three-term recurrence

Khang Tran
Truman State University, USA

Abstract

For any fixed positive integer n , we study the root distribution of a sequence of polynomials $H_m(z)$ satisfying the rational generating function

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^n}$$

where $A(z)$ and $B(z)$ are any polynomials in z with complex coefficients. We show that the roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on a specific fixed real algebraic curve for all large m .

1 Introduction

The sequence of polynomials $H_m(z)$, generated by the rational function $1/(1 + B(z)t + A(z)t^n)$, has the three-term recurrence relation of degree n

$$H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-n}(z) = 0 \quad (1)$$

and the initial conditions

$$H_m(z) = (-1)^m B^m(z), \quad 0 \leq m < n. \quad (2)$$

For the study of the root distribution of other sequences of polynomials that satisfy three-term recurrences, see [8, 14]. In [16], the author shows that in the three special cases when $n = 2, 3$, and 4 , the roots of $H_m(z)$ which satisfies $A(z) \neq 0$ will lie on the curve \mathcal{C} defined in Theorem 1, and are dense there as $m \rightarrow \infty$. This paper shows that for any fixed integer n , this result holds for all large m in the theorem below.

Theorem 1 *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^n}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. There is a constant $C = C(n)$ such that for all $m > C$, the roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on a fixed curve \mathcal{C} given by

$$\Im \frac{B^n(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq (-1)^n \Re \frac{B^n(z)}{A(z)} \leq \frac{n^n}{(n-1)^{n-1}}$$

and are dense there as $m \rightarrow \infty$.

This theorem holds when the numerator of the generating function is a monomial in t and z . For a general numerator, it appears, in an unpublished joint work with Robert Boyer, that the set of roots will approach \mathcal{C} and a possible finite set in the Hausdorff metric on the non-empty compact subsets. For more study of sequences of polynomials whose roots approach fixed curves, see [6, 7]. Other studies of the limits of zeros of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in z are given in [4, 5].

An important trinomial is $y^m - my + m - 1$. Its fundamental role in the study of inequalities is pointed out in both [3] and [13]. This paper shows that a “ θ -analogue” of this ($\theta = 0$) trinomial is fundamental for the study of polynomials generated by rational functions whose denominators are trinomials. Some further information about trinomials is available in [10]. It is also noteworthy that although there is no really concise formula for the discriminant of a general polynomial in terms of its coefficients, there is such a formula for the discriminant of a trinomial [11, pp. 406–407]. Here we develop, in the fashion of Ismail, a q -analogue of this discriminant formula. This plays a fundamental role in the determination of the curve \mathcal{C} .

Our main approach is to count the number of roots of $H_m(z)$ on the curve \mathcal{C} and show that this number equals the degree of this polynomial. This number of roots connects with the number of quotients of roots in t of the denominator $1 + B(z)t + A(z)t^n$ on a portion of the unit circle. The plot of these quotients when $n = 6$ and $m = 30$ is given in Figure 1. Although the curve \mathcal{C} depends on $A(z)$ and $B(z)$, it will be seen that this plot of the quotients is independent of these two polynomials.

For an example of Theorem 1, we consider the sequence of Chebyshev polynomials of the second kind, $U_n(x)$, where $B(z) = -2z$ and $A(z) = 1$. With $z = x + iy$, the curve \mathcal{C} is given by $xy = 0$ and $0 \leq 4(x^2 - y^2) \leq 4$ which is identical to the real interval $[-1, 1]$. In another example when $A(z) = z^3 + i$, $B(z) = z$, and $n = 5$, the curve \mathcal{C} is a portion of the curve given by the Cartesian equation

$$x(2x^6y + 6x^4y^3 - x^4 + 6x^2y^5 + 10x^2y^2 + 2y^7 - 5y^4) = 0.$$

The plot of the roots of $H_m(z)$ in the latter example is given in Figure 2.

2 Proof of the theorem

In this paper, we let $m = np + r$ where p and r are two positive integers with $0 \leq r < n$. We also let a and b be the degrees of the polynomials $A(z)$ and $B(z)$ respectively.

Lemma 2 *The degree of the polynomial $H_m(z)$ is at most*

$$\begin{cases} mb & \text{if } nb > a \\ pa + rb & \text{if } nb \leq a \end{cases}.$$

Proof The lemma follows from induction and the definition of $H_m(z)$ in (1) and (2).

For each z , we let $t_k = t_k(z)$, $0 \leq k < n$, be the roots of the denominator $1 + B(z)t + A(z)t^n$. We will focus our attention on the quotients of roots $q_k = t_k/t_0$, $0 \leq k < n$ because in Lemma 3 below, these quotients satisfy an equation which is independent of $A(z)$ and $B(z)$. See Figure 1 for the plot of these quotients when z runs through the roots of $H_m(z)$. This lemma also appears in a different unpublished joint work with Boyer and a proof is given for completeness.

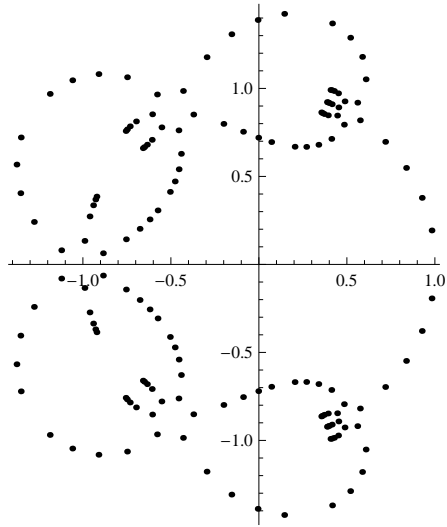


Figure 1: Distribution of the quotients of roots of the hexic denominator

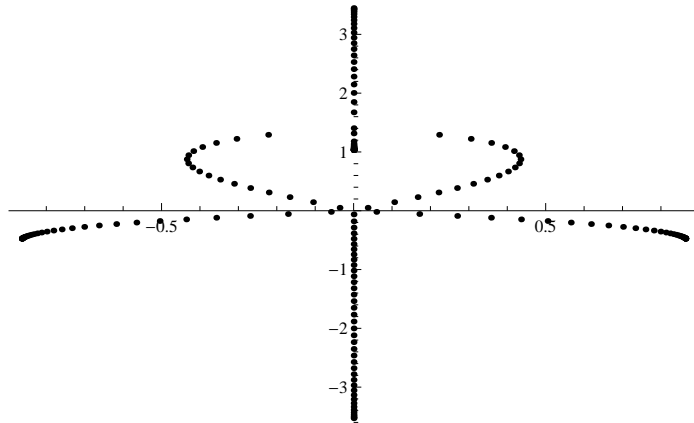


Figure 2: Distribution of the roots of $H_{200}(z)$ when $A(z) = z^3 + i$, $B(z) = z$, and $n = 5$

Lemma 3 Let z be a root of $H_m(z)$. If $q := q_1 = e^{2i\theta}$, $\theta \in \mathbb{R}$, then besides the two roots $e^{\pm i\theta}$, the remaining $n - 2$ roots of the polynomial

$$P_\theta(\zeta) = \zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta + \frac{\sin(n-1)\theta}{\sin \theta} \quad (3)$$

are $e^{-i\theta}q_2, \dots, e^{-i\theta}q_{n-1}$.

Proof Let $e_k(t_0, t_1, \dots, t_{n-1})$ be the k -th elementary symmetric polynomials in the variables t_0, t_1, \dots, t_{n-1} . Since t_0, t_1, \dots, t_{n-1} are the roots of $1 + B(z)t + A(z)t^n$, we have $e_k(t_0, t_1, \dots, t_{n-1}) = 0$ when $1 \leq k \leq n - 2$. Let $e_k := e_k(q_2, \dots, q_{n-1})$. We divide the equation $e_k(t_0, t_1, \dots, t_{n-1}) = 0$ by t_0^k , $1 \leq k \leq n - 2$, to obtain

$$\begin{cases} e_1 + q + 1 & = 0 \\ e_2 + e_1(q + 1) + q & = 0 \\ e_3 + e_2(q + 1) + e_1q & = 0 \\ e_4 + e_3(q + 1) + e_2q & = 0 \\ \vdots & \vdots \\ e_{n-2} + e_{n-3}(q + 1) + e_{n-4}q & = 0. \end{cases}$$

We solve this system of equations for e_1, \dots, e_{n-2} by substitution to get

$$e_k = (-1)^k(1 + q + q^2 + \dots + q^k)$$

with $1 \leq k \leq n - 2$. Assuming that $e_0 = 1$, the definition of e_k implies that q_2, q_3, \dots, q_{n-1} are the roots of the equation

$$\sum_{k=0}^{n-2} (-1)^k e_k \zeta^{n-k-2} = 0.$$

We multiply both side of this equation by $(1 - q)$ to obtain

$$(1 - q)\zeta^{n-2} + (1 - q^2)\zeta^{n-3} + (1 - q^3)\zeta^{n-4} + \dots + (1 - q^{n-1}) = 0.$$

Thus

$$\zeta^{n-2} + \zeta^{n-3} + \dots + 1 = q\zeta^{n-2} + q^2\zeta^{n-3} + \dots + q^{n-1}.$$

We add ζ^{n-1} to both sides to get

$$\frac{\zeta^n - 1}{\zeta - 1} = \frac{\zeta^n - q^n}{\zeta - q}. \quad (4)$$

This identity implies that

$$\zeta^n(1 - q) - (1 - q^n)\zeta + q(1 - q^{n-1}) = 0.$$

We divide both sides of this equation by $1 - q$ and note that

$$\begin{aligned} \frac{q^n - 1}{q - 1} &= e^{(n-1)i\theta} \frac{e^{ni\theta} - e^{-ni\theta}}{e^{i\theta} - e^{-i\theta}} \\ &= e^{(n-1)i\theta} \frac{\sin n\theta}{\sin \theta}. \end{aligned} \quad (5)$$

We obtain

$$\zeta^n - e^{(n-1)i\theta} \frac{\sin n\theta}{\sin \theta} \zeta + e^{ni\theta} \frac{\sin(n-1)\theta}{\sin \theta} = 0.$$

The lemma follows from a substitution of ζ by $e^{i\theta} \zeta$.

The quotients of the roots of a polynomial connect with the q -analogue of its discriminant. The q -discriminant, introduced by Mourad Ismail[15], of a general polynomial $P(x)$ of degree d_P and the leading coefficient a_P is

$$\text{Disc}_x(P; q) = a_P^{2n-2} q^{n(n-1)/2} \prod_{1 \leq i < j \leq d_P} (q^{-1/2} x_i - q^{1/2} x_j)(q^{1/2} x_i - q^{-1/2} x_j).$$

where x_i , $1 \leq i \leq d_P$, are the roots of $P(x)$. This q -discriminant is zero if and only if a quotient of roots x_i/x_j equals q . When q approaches 1, this q -discriminant becomes the ordinary discriminant $\text{Disc}_x P(x)$. This ordinary discriminant is also the resultant of $P(x)$ and its derivative. The resultant of any two polynomials $P(x)$ and $R(x)$ is given by

$$\begin{aligned} \text{Res}_x(P(x), R(x)) &= a_P^{d_R} \prod_{P(x_i)=0} R(x_i) \\ &= a_R^{d_P} \prod_{R(x_i)=0} P(x_i) \end{aligned} \tag{6}$$

where a_R and d_R are the leading coefficient and the degree of $R(x)$ respectively. For further information on the ordinary discriminant and resultant, see [1, 2, 9, 11, 12].

Ismail [15] showed that

Proposition 4 *The q -discriminant of a polynomial $P(x)$ of degree n with a lead coefficient γ is*

$$\text{Disc}_x(P; q) = (-1)^{n(n-1)/2} \gamma^{n-2} \prod_{i=1}^n (D_q P)(x_i)$$

where

$$(D_q P)(x) = \frac{P(x) - P(qx)}{x - qx}.$$

The lemma below also appears in an unpublished joint work with Boyer. A proof is given for completeness.

Lemma 5 *The q -discriminant of $D(t) = 1 + Bt + At^n$ is*

$$\text{Disc}_t(D(t); q) = \pm A^{n-2} \left(B^n \frac{q^{n-1}(1 - q^{n-1})^{n-1}}{(1 - q)^{n-1}} + (-1)^{n-1} \frac{(1 - q^n)^n}{(1 - q)^n} A \right).$$

Proof Proposition 4 gives

$$\text{Disc}_t(D(t); q) = \pm A^{n-2} \prod_{D(t)=0} (D_q D)(t)$$

where

$$\begin{aligned} (D_q D)(t) &= \frac{D(t) - D(qt)}{t - qt} \\ &= B + At^{n-1} \frac{1 - q^n}{1 - q}. \end{aligned} \quad (7)$$

Using the symmetric definition of resultant in (6), we write the product in $\text{Disc}_t(D(t); q)$ as

$$\text{Disc}_t(D(t); q) = \pm A^{n-1} \left(\frac{1 - q^n}{1 - q} \right)^n \prod_{(D_q D)(t)=0} D(t). \quad (8)$$

From the definition of $D(t)$ and the formula of $(D_q D)(t)$ in (7), these two polynomials are connected by

$$\frac{1 - q^n}{1 - q} D(t) = t(D_q D)(t) + tB \left(\frac{1 - q^n}{1 - q} - 1 \right) + \frac{1 - q^n}{1 - q}.$$

Thus (8) gives

$$\begin{aligned} \text{Disc}_t(D(t); q) &= \pm A^{n-1} \left(\frac{1 - q^n}{1 - q} \right) \prod_{(D_q D)(t)=0} \left(tB \left(\frac{1 - q^n}{1 - q} - 1 \right) + \frac{1 - q^n}{1 - q} \right) \\ &= \pm A^{n-2} B^n \frac{q^n (1 - q^{n-1})^n}{(1 - q)^n} (D_q D) \left(-\frac{1 - q^n}{Bq(1 - q^{n-1})} \right) \\ &= \pm A^{n-2} \left(B^n \frac{q^{n-1} (1 - q^{n-1})^{n-1}}{(1 - q)^{n-1}} + (-1)^{n-1} \frac{(1 - q^n)^n}{(1 - q)^n} A \right). \end{aligned}$$

From Lemma 4, we see that in the case $q = 1$ all the points z such that $\text{Disc}_t D(t) = 0$ belong to the curve \mathcal{C} . Thus we only need to consider the case $\text{Disc}_t D(t) \neq 0$, i.e., all the roots t_0, t_1, \dots, t_{n-1} are distinct. The polynomial $P_\theta(\zeta)$ in Lemma 3 plays an important role in the proof of Theorem 1. In particular, instead of counting the number of roots of $H_m(z)$ on \mathcal{C} , we will count the number of real roots of a function of θ where $q_1 = e^{2i\theta}$. This function is given by the lemma below.

Lemma 6 *Let z be a point on \mathcal{C} such that $q_1 = e^{2i\theta}$, $\theta \in \mathbb{R}$ and $A(z) \neq 0$. Then z is a root of $H_m(z)$ if and only if*

$$h(\theta) := \sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)} = 0$$

where $\zeta_0, \dots, \zeta_{n-1}$ are the roots of

$$P_\theta(\zeta) = \zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta + \frac{\sin(n-1)\theta}{\sin \theta}.$$

Proof By partial fractions, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} H_m(z)t^m &= \frac{1}{A(z)(t-t_0)(t-t_2)\cdots(t-t_{n-1})} \\
&= \frac{1}{A(z)} \sum_{k=0}^{n-1} \frac{1}{t-t_k} \prod_{l \neq k} \frac{1}{t_k-t_l} \\
&= \frac{1}{A(z)} \sum_{k=0}^{n-1} \frac{1}{t_k^{m+1}} \prod_{l \neq k} \frac{1}{t_k-t_l} t^m.
\end{aligned}$$

Thus $H_m(z) = 0$ is equivalent to

$$\sum_{k=0}^{n-1} \frac{1}{t_k^{m+1}} \prod_{l \neq k} \frac{1}{t_k-t_l} = 0.$$

We multiply the equation by t_0^{m+n} and let $q_k = t_k/t_0$ to obtain

$$\sum_{k=0}^{n-1} \frac{1}{q_k^{m+1}} \prod_{l \neq k} \frac{1}{q_k-q_l} = 0. \tag{9}$$

Let $\zeta_k = e^{-i\theta} q_k$, $0 \leq k < n$. Lemma 3 implies that $\zeta_0, \dots, \zeta_{n-1}$ are the roots of

$$P_\theta(\zeta) = \zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta + \frac{\sin(n-1)\theta}{\sin \theta}$$

where $\zeta_0 = e^{-i\theta}$ and $\zeta_1 = e^{i\theta}$. We multiply (9) by $e^{(m+n)i\theta}$ and obtain

$$\sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)} = 0.$$

Since $P_\theta(\zeta)$ is a real polynomial in ζ , the function $h(\theta)$ is a real-valued function of θ by the symmetric reduction. Each real root θ of $h(\theta)$ yields some roots z of $H_m(z)$ on \mathcal{C} . For example, if $\theta = \pi/n$ is a root of $h(\theta)$ then a root of $B(z)$ will also be a root of $H_m(z)$ on \mathcal{C} since, from Lemma 5,

$$(-1)^n \frac{B^n(z)}{A(z)} = \frac{(1-q^n)^n}{(1-q)q^{n-1}(1-q^{n-1})^{n-1}} \tag{10}$$

where $q = e^{2i\theta}$. The lemma below gives the number of roots of $H_m(z)$ in the case of $\theta = \pi/n$.

Lemma 7 *If $m = np + r$ with $0 \leq r < n$ then the polynomial $B^r(z)$ divides $H_m(z)$.*

Proof The generating function of $H_m(z)$ is

$$\begin{aligned}
\frac{1}{1+B(z)t+A(z)t^n} &= \sum_{k=0}^{\infty} (-1)^k (B(z)t + A(z)t^n)^k \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^k (-1)^k \binom{k}{i} t^{n(k-i)+i} A^{k-i}(z) B^i(z).
\end{aligned}$$

The lemma follows from the power of t in the double summation above.

Our main goal is to show that the number of real roots θ of $h(\theta)$ on the interval $[0, \pi/n)$ is p when m is large. See Figure 1 for a plot of these p roots on the arc $q = e^{2i\theta}$. Assuming this fact, which will be proved later, we provide an argument which shows that the roots of $H_m(z)$ lie on \mathcal{C} . From Lemma 5, if q is a quotient of two roots in t of $D(t, z)$ then

$$(-1)^n B^n(z)(1-q)q^{n-1}(1-q^{n-1})^{n-1} = (1-q^n)^n A(z). \quad (11)$$

If θ is a root of $h(\theta)$ on the interval $[0, \pi/n)$ then, with $q = e^{2i\theta}$, each solution in z of (11) is a root of $H_m(z)$. From (5), we write the right side of (10) in θ and see that it satisfies

$$0 \leq \frac{\sin^n n\theta}{\sin \theta \sin^{n-1}(n-1)\theta} \leq \frac{n^n}{(n-1)^{n-1}} \quad (12)$$

since $\sin n\theta / \sin \theta \leq n$ and $\sin n\theta / \sin(n-1)\theta \leq n/(n-1)$. Thus each solution z of (11) belongs to \mathcal{C} . In the case $nb > a$, the equation (11) shows that each value of q gives nb solutions of z counting multiplicities. Thus we have at least nbp solutions of $H_m(z)$ on \mathcal{C} . Lemma 7 gives us at least rb more solutions of $H_m(z)$ on \mathcal{C} . Thus the number of roots of $H_m(z)$ on \mathcal{C} is at least $nbp + rb = mb$. With a similar argument, when $nb \leq a$, the number of roots of $H_m(z)$ on \mathcal{C} is at least $pa + rb$. This number of roots equals the degree of $H_m(z)$ in Lemma 2. Hence all the roots of $H_m(z)$ lie on \mathcal{C} .

Let $\zeta_0, \dots, \zeta_{n-1}$ be the roots of $P_\theta(\zeta)$ for $\theta \in [0, \pi/n)$. We will show that the number of real roots of

$$h(\theta) = \sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)}, \quad (13)$$

is p where $m = np + r$, $0 \leq r < n$. With the convention that $\sin n\theta / \sin \theta = n$ when $\theta = 0$, we have the lemma below.

Lemma 8 *If $0 \leq \theta < \pi/n$ then, besides the two roots $e^{\pm i\theta}$, the remaining $n-2$ roots of the polynomial*

$$P_\theta(\zeta) = \zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta + \frac{\sin(n-1)\theta}{\sin \theta}$$

lie outside the closed unit disk, i.e., $|\zeta| > 1$.

Proof We replace ζ by $\zeta e^{-i\theta}$ in $P_\theta(\zeta)$ and then rewrite (4) as

$$\frac{\zeta^n - 1}{\zeta - 1} = \frac{q^n - 1}{q - 1}.$$

From (5), the polynomial map

$$f(z) = \frac{z^n - 1}{z - 1}$$

bijectionally maps the arc $q = e^{2i\theta}$, $-\pi/n < \theta < \pi/n$, to the biggest loop such as the one in in Figure 3 when $n = 6$. Thus by the Argument Principle Theorem it maps the open unit disk into the interior of this loop. Hence the equation $f(\zeta) = f(q)$ does not have a solution $|\zeta| \leq 1$ except $\zeta = q$.

We will find $p+1$ values for θ where the sign of $h(\theta)$ alternates when m is large and then apply the Intermediate Value Theorem to obtain p real roots of $h(\theta)$. Lemma 8 and the definition of $h(\theta)$ in (13) imply that if θ does not approach π/n as $m \rightarrow \infty$ and $P'_\theta(\zeta_k) \neq 0$, the sign of $h(\theta)$ depends on the two summands when $k=0$ and $k=1$ when m is large. The second condition, $P'_\theta(\zeta_k) \neq 0$, is given below.

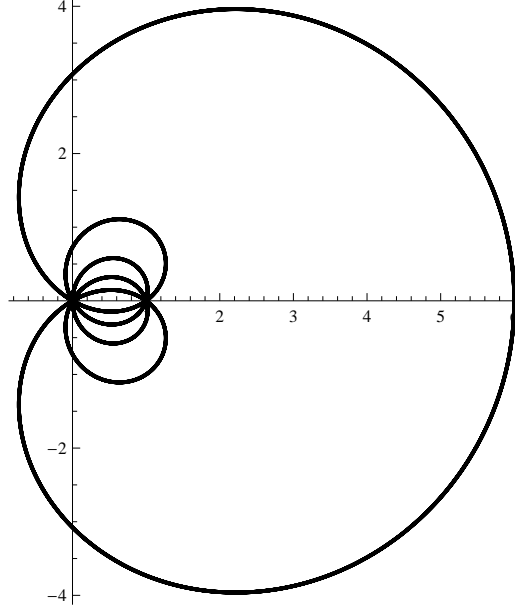


Figure 3: The map $h(z)$ on the unit circle when $n = 6$

Lemma 9 *If $0 < \theta < \pi/n$ then the roots of the polynomial*

$$P_\theta(\zeta) = \zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta + \frac{\sin(n-1)\theta}{\sin \theta}$$

are distinct. When $\theta = 0$, besides the double root at $\zeta = 1$, the remaining $n - 2$ roots are distinct.

Proof If ζ is a root of $P_\theta(\zeta)$ then

$$\begin{aligned} \zeta P'_\theta(\zeta) &= n\zeta^n - \frac{\sin n\theta}{\sin \theta} \zeta \\ &= (n-1) \frac{\sin n\theta}{\sin \theta} \zeta - n \frac{\sin(n-1)\theta}{\sin \theta}. \end{aligned}$$

By the symmetry in the product definition of $\text{Res}_\zeta(P_\theta(\zeta), P'_\theta(\zeta))$ in (6), the roots of $P_\theta(\zeta)$ are distinct if and only if

$$P_\theta\left(\frac{n \sin(n-1)\theta}{(n-1) \sin n\theta}\right) \neq 0.$$

With the definition of $P_\theta(\zeta)$, this condition becomes

$$\frac{\sin^n n\theta}{\sin \theta \sin^{n-1}(n-1)\theta} \neq \frac{n^n}{(n-1)^{n-1}}.$$

From (12), the equation occurs only when $\theta = 0$. In this case, besides the double root at 1, the remaining roots of $P_0(\zeta) = \zeta^n - n\zeta + (n-1)\zeta$ are distinct.

We recall that $m = np + r$, $0 \leq r < n$. The lemma below shows how $h(\theta)$ alternates sign for θ not approaching π/n as $m \rightarrow \infty$.

Lemma 10 *Let γ be a fixed small real number and*

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}, \quad \text{where} \quad l = h + \frac{r}{n}$$

and h is a non-negative integer. If

$$\gamma m < l < \frac{m}{n},$$

then the sign of $h(\theta)$ is $(-1)^{p-h+1}$ when m is large.

Proof If $2 \leq k \leq n-1$ then Lemmas 8 and 9 imply that $|\zeta_k| > 1 + \epsilon$ and $|P'_\theta(\zeta_k)| > \epsilon$ for some small $\epsilon > 0$ which does not depend on m . Thus

$$\frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)}$$

approaches 0 exponentially when m approaches ∞ .

Since $\zeta_0 = e^{-i\theta}$ and $\zeta_1 = e^{i\theta}$, the sum of the first two terms of (13) when $k=0$ and $k=1$ is

$$\frac{2\Re(e^{i(m+1)\theta} P'_\theta(e^{i\theta}))}{|P'_\theta(e^{i\theta})|^2}.$$

With fact that $P'_\theta(\zeta) = n\zeta^{n-1} - \sin n\theta / \sin \theta$, this sum becomes

$$\frac{2n \cos(m+n)\theta - 2 \cos(m+1)\theta \sin n\theta / \sin \theta}{|P'_\theta(e^{i\theta})|^2}. \quad (14)$$

When $\theta = \pi/n - l\pi/m = (p-h)\pi/m$ we have $\cos m\theta = (-1)^{p-h}$. The numerator of (14) becomes

$$(-1)^{p-h+1} \frac{2}{\sin \theta} (\cos \theta \sin n\theta - n \cos n\theta \sin \theta). \quad (15)$$

Since $\theta = (p-h)\pi/m > 0$ where p and h are non-negative integers, we have $\theta \geq \pi/m > 1/m$. The function $\cos \theta \sin n\theta - n \cos n\theta \sin \theta$ is positive and increasing on $[0, \pi/n]$ since its derivative is $(n^2 - 1) \sin \theta \sin n\theta \geq 0$. Thus the condition $\theta > 1/m$ implies

$$\cos \theta \sin n\theta - n \cos n\theta \sin \theta > \cos \frac{1}{m} \sin \frac{n}{m} - n \cos \frac{n}{m} \sin \frac{1}{m}.$$

When m is large, the right side is close to $n^3/2m^3$ which is larger than all of the exponentially decaying terms where $2 \leq k < n$. Thus the sign of $h(\theta)$ is $(-1)^{p-h+1}$ by (15).

We next consider the case when θ approaches π/n .

Lemma 11 *Let $\theta = \pi/n - l\pi/m$ where $l \in \mathbb{R}^+$. There is a sufficiently small δ such that if $l < \delta m$ then*

$$P_\theta(\zeta) = \zeta^n + 1 - \frac{nl\pi}{m \sin(\pi/n)} \zeta + \frac{\cos(\pi/n)}{\sin(\pi/n)} \cdot \frac{nl\pi}{m} + O_n\left(\frac{l^2}{m^2}\right), \quad (16)$$

$$P'_\theta(\zeta) = n\zeta^{n-1} - \frac{\sin(nl\pi/m)}{\sin(\pi/n)} + O_n\left(\frac{l^2}{m^2}\right), \quad (17)$$

$$\zeta_k = e_k \left(1 + \frac{l\pi(\cos(\pi/n) - e_k)}{m \sin(\pi/n)}\right) + O_n\left(\frac{l^2}{m^2}\right). \quad (18)$$

Proof The identity (16) follows from the definition of $P_\theta(\zeta)$ in (3) and basic asymptotic computations. We differentiate $P_\theta(\zeta)$ and get

$$\begin{aligned} P'_\theta(\zeta) &= n\zeta^{n-1} - \frac{\sin n\theta}{\sin \theta} \\ &= n\zeta^{n-1} - \frac{\sin(nl\pi/m)}{\sin(\pi/n)} + O_n\left(\frac{l^2}{m^2}\right). \end{aligned}$$

Suppose $\zeta_k = e_k + \epsilon$ where $\epsilon \in \mathbb{C}$ and $e_k = e^{(2k-1)i\pi/n}$, $0 \leq k \leq n-1$. The facts that $P_\theta(\zeta_k) = 0$ and (16) imply that

$$-\frac{n\epsilon}{e_k} + \frac{nl\pi}{m \sin \pi/n} \left(\cos \frac{\pi}{n} - e_k \right) + O_n\left(\epsilon^2 + \frac{l\epsilon}{m} + \frac{l^2}{m^2}\right) = 0.$$

This gives

$$\begin{aligned} \epsilon &= \frac{l\pi e_k}{m \sin \pi/n} \left(\cos \frac{\pi}{n} - e_k \right) + O_n\left(\epsilon^2 + \frac{l\epsilon}{m} + \frac{l^2}{m^2}\right) \\ &= \frac{l\pi e_k}{m \sin \pi/n} \left(\cos \frac{\pi}{n} - e_k \right) + O_n\left(\frac{l^2}{m^2}\right). \end{aligned}$$

Thus

$$\zeta_k = e_k \left(1 + \frac{l\pi(\cos \pi/n - e_k)}{m \sin \pi/n} \right) + O_n\left(\frac{l^2}{m^2}\right).$$

Lemma 12 *Let $\delta > 0$ be a fixed real number and*

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}, \quad \text{where} \quad l = h + \frac{r}{n}$$

and h is a non-negative integer. There is a sufficiently small γ such that if $\delta\sqrt{m} < l < \gamma m$ the sign of

$$\sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)}$$

is $(-1)^{p-h+1}$ when m is large.

Proof From (18), we can choose γ small enough so that

$$\begin{aligned} |\zeta_k| &> 1 + \frac{l\pi(\cos \pi/n - \cos(2k-1)\pi/n)}{2m \sin \pi/n} \\ &> 1 + \frac{\delta\pi(\cos \pi/n - \cos(2k-1)\pi/n)}{2\sqrt{m} \sin \pi/n}. \end{aligned}$$

Thus if $k \neq 0, 1$ then the inequality above and (19) imply that

$$\frac{1}{|\zeta_k^{m+1} P'_\theta(\zeta_k)|}$$

approaches 0 when $m \rightarrow \infty$. From (14), the sum of two terms when $k = 0, 1$ is

$$\frac{2n \cos(m+n)\theta - 2 \cos(m+1)\theta \sin n\theta / \sin \theta}{|P'_\theta(e^{i\theta})|^2}.$$

When γ is small, the quantity $\cos(m+1)\theta \sin n\theta / \sin \theta$ is small since

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}.$$

Also $|P'_\theta(e^{i\theta})|^2$ is close to n^2 by (19). Thus the sign of (13) is determined by the sign of

$$\cos(m\theta + n\theta) = (-1)^{p-h} \cos n\theta,$$

which is $(-1)^{p-h+1}$, when m is large.

Lemma 13 *Let*

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}, \quad \text{where} \quad l = h + \frac{r}{n}$$

and h is a non-negative integer. There is a sufficiently small δ so that if $1 \leq l < \delta\sqrt{m}$ the sign of

$$h(\theta) = \sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)}$$

is $(-1)^{p-h+1}$ when m is large.

Proof With a sufficiently small δ , the equation (18) gives

$$\begin{aligned} \zeta_k^{m+1} &= e_k^{m+1} \left(1 + \frac{l\pi(\cos \pi/n - e_k)}{m \sin \pi/n} \right)^m \left(1 + O_n \left(\frac{l^2}{m} \right) \right) \\ &= e_k^{m+1} e^{l\pi(\cos \pi/n - e_k)/\sin \pi/n} \left(1 + O_n \left(\frac{l^2}{m} \right) \right). \end{aligned}$$

Also equations (17) and (18) imply that

$$P'_\theta(\zeta_k) = n e_k^{n-1} \left(1 + O_n \left(\frac{l}{m} \right) \right). \quad (19)$$

We combine (19) with the estimation of ζ_k^{m+1} to get

$$\zeta_k^{m+1} P'_\theta(\zeta_k) = n e_k^{m+n} e^{l\pi(\cos \pi/n - e_k)/\sin \pi/n} \left(1 + O_n \left(\frac{l^2}{m} \right) \right). \quad (20)$$

According to (20), the summation of the two terms of $h(\theta)$ when $k = 0$ and $k = 1$ is

$$\frac{2}{n} \Re(e_k^{m+n} e^{-l\pi i}) = \frac{2}{n} \Re(e^{\pi i(m-ln+n)/n} \left(1 + O_n \left(\frac{l^2}{m} \right) \right)).$$

With $l = h + r/n$, the expression becomes

$$\frac{(-1)^{p-h+1} 2}{n} \left(1 + O_n \left(\frac{l^2}{m} \right) \right). \quad (21)$$

We now consider the remaining terms of $h(\theta)$. From (20), the summation of these terms is bounded by

$$\frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{e^{\pi(\cos \pi/n - \cos(2k-1)\pi/n) / \sin \pi/n}} \left(1 + O_n \left(\frac{l^2}{m} \right) \right).$$

We use computer algebra to check that if $n < 90$ then

$$\frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{e^{\pi(\cos \pi/n - \cos(2k-1)\pi/n) / \sin \pi/n}} < \frac{2}{n}.$$

From (21), the lemma holds when $n < 90$.

We now consider the case $n \geq 90$. From (20), the sign of $h(\theta)$ is the same as that of

$$\frac{1}{n} \sum_{k=0}^{n-1} e_k^{-m-n} e^{l\pi e_k / \sin \pi/n}.$$

We can write this summation as

$$\frac{1}{n} \sum_{k=0}^{n-1} e_k^{-m-n} e^{lne_k} + \epsilon$$

where ϵ is a small number. The Taylor expansion gives

$$\frac{1}{n} \sum_{k=0}^{n-1} e_k^{-m-n} e^{lne_k} = \frac{1}{n} \sum_{k=0}^{n-1} e_k^{-m-n} \sum_{j=0}^{\infty} \frac{(lne_k)^j}{j!}.$$

With the fact that

$$\sum_k e_k^a = \begin{cases} 0 & \text{if } n \nmid a \\ n(-1)^{a/n} & \text{if } n|a \end{cases},$$

the double summation becomes

$$\sum_{j=0}^{\infty} (-1)^{j+p+1} \frac{(ln)^{jn+r}}{(jn+r)!}.$$

With $l = h + r/n$, this summation is

$$\sum_{j=0}^{\infty} (-1)^{j+p+1} \frac{(nh+r)^{nj+r}}{(nj+r)!}. \quad (22)$$

By taking the natural logarithm, we leave it to the reader to check that the absolute value of the summand increases from $j = 0$ to $j = h$ and then decreases when $j > h$. Thus the alternating signs imply that

$$\left| \sum_{j=0}^{h-1} (-1)^{j+p+1} \frac{(nh+r)^{nj+r}}{(nj+r)!} \right| \leq \frac{(nh+r)^{n(h-1)+r}}{(n(h-1)+r)!},$$

and

$$\left| \sum_{j=h+1}^{\infty} (-1)^{j+p+1} \frac{(nh+r)^{nj+r}}{(nj+r)!} \right| \leq \frac{(nh+r)^{n(h+1)+r}}{(n(h+1)+1)!}.$$

Since

$$\frac{(nh+r)^{n(h-1)+r}}{(n(h-1)+r)!} + \frac{(nh+r)^{n(h+1)+r}}{(n(h+1)+1)!} < \frac{(nh+r)^{nh+r}}{(nh+r)!},$$

the sign of (22) is $(-1)^{h+p+1}$. The lemma follows.

Lemma 14 *Let*

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}, \quad \text{where} \quad l = h + \left\{ \frac{m}{n} \right\},$$

h is an non-negative integer, and $l \geq 1$. The sign of

$$\sum_{k=0}^{n-1} \frac{1}{\zeta_k^{m+1} P'_\theta(\zeta_k)}$$

is $(-1)^{\lfloor m/n \rfloor - h + 1}$ when m is large.

Proof This lemma follows from Lemmas 13, 12, and 10.

Lemma 15 *The function $h(\theta)$ satisfies $h(0^+) < 0$.*

Proof

Equation (14) in the proof of Lemma 10 shows that the main term of $h(\theta)$ is

$$\frac{2n \cos(m+n)\theta - 2 \cos(m+1)\theta \sin n\theta / \sin \theta}{|P'_\theta(e^{i\theta})|^2}.$$

The numerator of this is

$$\begin{aligned} & 2n \left(1 - \frac{(m+n)^2 \theta^2}{2} + O(m^4 \theta^4) \right) \\ & - 2n \left(1 - \frac{(m+1)^2 \theta^2}{2} + O(m^4 \theta^4) \right) (1 + O(\theta^2)). \end{aligned}$$

The claim follows.

Lemma 16 *The sign of*

$$h\left(\frac{\pi^-}{n}\right)$$

is $(-1)^{p+1}$.

Proof Let $m = pn + q$, $0 \leq q < n$. We consider equation (20)

$$\zeta_k^{m+1} P'_\theta(\zeta_k) = n e_k^{m+n} e^{l\pi(\cos \pi/n - e_k)/\sin \pi/n} \left(1 + O_n \left(\frac{l^2}{m} \right) \right)$$

where $\theta = \pi/n - l\pi/m$. By using the first n terms of the Taylor expansion for l sufficiently small, we see that the sign of $h(\theta)$ is equal to the sign of

$$\frac{1}{n e^{l\pi \cot \pi/n}} \sum_{k=0}^{n-1} e_k^{-(m+n)} \sum_{j=0}^{n-1} \frac{1}{j!} \left(\frac{l\pi e_k}{\sin \pi/n} \right)^j$$

if this quantity is nonzero. By the fact that

$$\sum_k e_k^a = \begin{cases} 0 & \text{if } n \nmid a \\ n(-1)^{a/n} & \text{if } n|a \end{cases},$$

the expression becomes

$$\frac{1}{e^{l\pi \cot \pi/n}} \frac{(-1)^{p+1}}{r!}.$$

The lemma follows.

We now provide an argument that shows the function $h(\theta)$ has p real roots on the interval $[0, \pi/n)$. Lemmas 14, 15, and 16 show that the sign of $h(\theta)$ alternates when θ varies among $p + 1$ values

$$\theta = \frac{\pi}{n} - \frac{l\pi}{m}$$

where $l = h + \{m/n\}$, $1 \leq h \leq p - 1$, and

$$\theta = 0^+, \frac{\pi^-}{n}.$$

The claim then follows from the Intermediate Value Theorem.

Each real root $\theta \in [0, \pi/n)$ of $h(\theta)$ yields a certain number of roots z of $H_m(z)$ by Lemma 6 and (11). If z is such a root, it lies on the curve \mathcal{C} since the imaginary part of $B^n(z)/A(z)$ is zero and $0 \leq (-1)^n \Re(B^n(z)/A(z)) \leq n^n/(n-1)^{n-1}$ by (5), (10), and (12). The number of roots of $H_m(z)$ on \mathcal{C} is at least the degree of $H_m(z)$ by the argument after the proof of Lemma 7.

We end this paper with an argument which shows the density of the roots of $H_m(z)$ on \mathcal{C} . We first notice that the $p + 1$ values of θ mentioned above are dense on the interval $[0, \pi/n]$ when $m \rightarrow \infty$. From (10), the rational map $(-1)^n B^n(z)/A(z)$ maps an open neighborhood of a point on \mathcal{C} to an open set which contains a solution θ of $h(\theta)$ where $q = e^{2i\theta}$. Lemma 6 shows that there is a solution of $H_m(z)$ in U . The density of the roots of $H_m(z)$ on \mathcal{C} follows.

References

- [1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

- [2] T. M. Apostol, The resultants of the cyclotomic polynomials $F_m(ax)$ and $F_n(bx)$, *Math. Comp.* 29 (1975), 1–6.
- [3] E. F. Beckenbach, R. E. Bellman, *Inequalities*, Springer, Berlin, 1961.
- [4] S. Beraha, J. Kahane, N. J. Weiss, Limits of zeroes of recursively defined polynomials, *Proc. Nat. Acad. Sci. U.S.A.* 72 (1975), no. 11, 4209.
- [5] S. Beraha, J. Kahane, N. J. Weiss, Limits of zeros of recursively defined families of polynomials, *Studies in foundations and combinatorics*, pp. 213–232, *Adv. in Math. Suppl. Stud.*, 1, Academic Press, New York-London, 1978.
- [6] R. Boyer, W. M. Y. Goh, On the zero attractor of the Euler polynomials, *Adv. in Appl. Math.* 38 (2007), no. 1, 97–132.
- [7] R. Boyer, W. M. Y. Goh, Polynomials associated with partitions: asymptotics and zeros, *Special functions and orthogonal polynomials*, 33–45, *Contemp. Math.*, 471, Amer. Math. Soc., Providence, RI, 2008.
- [8] M. Charalambides, G. Csordas, The distribution of zeros of a class of Jacobi polynomials, *Proc. Amer. Math. Soc.* 138 (2010), no. 12, 4345–4357.
- [9] K. Dilcher and K. B. Stolarsky, Resultants and discriminants of Chebyshev and related polynomials, *Trans. Amer. Math. Soc.* 357 (2005), no. 3, 965–981.
- [10] K. Dilcher, J. D. Nulton, K. B. Stolarsky, The zeros of a certain family of trinomials, *Glasgow Math. J.* 34 (1992), 55–74.
- [11] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Determinants, Resultants, and Multidimensional Determinants*, Birkhäuser Boston, 1994.
- [12] J. Giske and M. E. H. Ismail, Resultants of Chebyshev polynomials, *Z. Anal. Anwend.* 27 (2008), no. 4, 499–508.
- [13] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1964.
- [14] M. X. He, E. B. Saff, The zeros of Faber polynomials for an m -cusped hypocycloid, *J. Approx. Theory* 78 (1994), no. 3, 410–432.
- [15] M. E. H. Ismail, Difference equations and quantized discriminants for q -orthogonal polynomials, *Adv. in Appl. Math.* 30 (2003), no. 3, 562–589.
- [16] K. Tran, Connections between discriminants and the root distribution of polynomials with rational generating function, *J. Math. Anal. Appl.* 410 (2014), 330–340.