1. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

**Solution:**

*Proof.* (contrapositive)

Suppose that $x \in \mathbb{R}$ such that $x \leq -1$. Then

$$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1).$$

Since $x \leq -1$, we have that $x \leq 0$,
$$x + 1 \leq -1 + 1, \text{ or } x + 1 \leq 0,$$
and $x - 1 \leq -1 - 1 = -2$, so $x - 1 \leq 0$. Since the product of three negative numbers is negative, we have that $x^3 - x \leq 0$. \hfill $\square$

2. The product of an irrational number and a nonzero rational number is irrational.

**Solution:**

*Proof.* (contradiction) Suppose to the contrary, that there exist an irrational number $a$ and a nonzero rational number $b$ whose product is rational. Since $ab$ is rational, we may write

$$ab = \frac{m}{n},$$

where $m, n \in \mathbb{Z}$ and $n \neq 0$. And, since $a$ is a nonzero rational number, we can write

$$a = \frac{k}{l},$$

where $k, l \in \mathbb{Z}$, $k \neq 0$, and $l \neq 0$. Then

$$\frac{k}{l}(b) = \frac{m}{n},$$

and multiplying both sides of the equation by $\frac{l}{k}$ gives

$$b = \frac{m}{n} \frac{l}{k} = \frac{ml}{nk}.$$

Since $ml$ and $nk$ are integers and $nk \neq 0$, we have that $b$ is rational, a contradiction. \hfill $\square$
3. If \( a \equiv b \pmod{n} \), then \( \gcd(a, n) = \gcd(b, n) \).

**Solution:**

**Proof.** (direct) Suppose that \( a \equiv b \pmod{n} \). Then, \( n \mid (a - b) \), so there exists an integer \( n \) such that \( a - b = kn \). Let \( d = \gcd(a, n) \). Then \( d \mid a \) and \( d \mid n \), so there exist integers \( x \) and \( y \) such that \( a = dx \) and \( n = dy \). Substituting this into \( a - b = kn \) gives \( dx - b = kdy \), or

\[
b = dx - dky = d(x - ky),
\]

so since \( x - ky \in \mathbb{Z}, d \mid b \). Since \( d \mid n \) and \( d \mid b \), \( \gcd(b, n) \geq d \), or \( \gcd(b, n) \geq \gcd(a, n) \).

Now, let \( e = \gcd(b, n) \). Then, \( e \mid b \) and \( e \mid n \), so there exist integers \( w \) and \( z \) such that \( b = ew \) and \( n = ez \). Substituting this into \( a - b = kn \) gives \( a - ew = k(ez) \), or

\[
a = ekz + ew = e(kz + w),
\]

and so, since \( kz + w \in \mathbb{Z}, d \mid a \). Since \( e \mid a \) and \( e \mid n \), \( e \leq \gcd(a, n) \), or \( \gcd(b, n) \leq \gcd(a, n) \).

Since we have \( \gcd(a, n) \leq \gcd(b, n) \) and \( \gcd(a, n) \geq \gcd(b, n) \), it must be true that \( \gcd(a, n) = \gcd(b, n) \). \( \square \)

4. Suppose \( a \in \mathbb{Z} \). If \( a^2 \) is not divisible by 4, then \( a \) is odd.

**Solution:**

**Proof.** (contrapositive) Suppose that \( a \) is an even integer. Then \( a = 2k \) for some integer \( k \). Therefore, \( a^2 = (2k)^2 = 4k^2 \), so since \( k^2 \) is an integer, \( a^2 \) is divisible by 4. \( \square \)

5. If \( a \in \mathbb{Z} \) and \( a \equiv 1 \pmod{5} \), then \( a^2 \equiv 1 \pmod{5} \).

**Solution:**

**Proof.** (direct)

Let \( a \in \mathbb{Z} \) and \( a \equiv 1 \pmod{5} \). Then \( 5 \mid (a - 1) \), so there exists an integer \( n \) such that \( a - 1 = 5n \). Therefore,

\[
a = 5n + 1
\]
\[
a^2 = (5n + 1)^2
\]
\[
= 25n^2 + 10n + 1
\]
\[
= 5(5n^2 + 2n) + 1.
\]

So,

\[
a^2 - 1 = 5(n^2 + 2n).
\]

Since \( n^2 + 2n \) is an integer, \( 5 \mid (a^2 - 1) \), or \( a^2 \equiv 1 \pmod{5} \). \( \square \)
6. If $a$ and $b$ are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

**Solution:** Side work: I’m going to do some algebra to see if something comes to mind.

\[
\begin{align*}
    a + b & \geq 2\sqrt{ab} \\
    (a + b)^2 & \geq (2\sqrt{ab})^2 \\
    a^2 + 2ab + b^2 & \geq 4ab \\
    a^2 - 2ab + b^2 & \geq 0 \\
    (a - b)^2 & \geq 0
\end{align*}
\]

So, two approaches to this proof come to mind: direct or by contradiction.

**Proof.** (direct) Let $a$ and $b$ be positive real numbers. Then $(a - b)^2 \geq 0$. Therefore,

\[
    a^2 - 2ab + b^2 \geq 0.
\]

Adding $4ab$ to both sides gives

\[
\begin{align*}
    a^2 + 2ab + b^2 & \geq 4ab \\
    (a + b)^2 & \geq 4ab.
\end{align*}
\]

Since $a$, $b$, and $(a + b)^2$ are all positive, we can take the square root of both sides, obtaining

\[
a + b \geq \sqrt{4ab} = 2\sqrt{ab}.
\]

Therefore, $a + b \geq 2\sqrt{ab}$ for any positive real numbers $a$ and $b$. \hfill \square

**Proof.** (contradiction) Suppose to the contrary that $a$ and $b$ are positive real numbers such that $a + b < 2\sqrt{ab}$. Then, since $(a + b)^2$ and $2\sqrt{ab}$ are nonnegative, we can take the square of both sides, and we have

\[
\begin{align*}
    (a + b)^2 & < (2\sqrt{ab})^2 \\
    a^2 + 2ab + b^2 & < 4ab \\
    a^2 - 2ab + b^2 & < 0 \\
    (a - b)^2 & < 0,
\end{align*}
\]

a contradiction. Therefore, $a + b \geq 2\sqrt{ab}$ for any positive real numbers $a$ and $b$. \hfill \square
7. Let $a \in \mathbb{Z}$. If $(a + 1)^2 - 1$ is even, then $a$ is even.

**Solution:**

*Proof.* (contrapositive) Suppose that $a$ is an odd integer. Then $a = 2k + 1$ for some integer $k$. So

\[
(a + 1)^2 - 1 = (2k + 2)^2 - 1 \\
= 4k^2 + 8k + 3 \\
= 4k^2 + 8k + 2 + 1 \\
= 2(2k^2 + 4k + 1) + 1.
\]

Since $2k^2 + 4k + 1$ is an integer, $(a + 1)^2 - 1$ is odd. \[\square\]

8. Let $a, b \in \mathbb{Z}$. If $a \geq 2$, then either $a \nmid b$ or $a \nmid (b + 1)$.

**Solution:**

*Proof.* (contradiction) Suppose to the contrary, that there exist integers $a$ and $b$ such that $a \geq 2$ and both $a \mid b$ and $a \mid (b + 1)$. Since $a \mid b$, then $b = ax$ for some integer $x$. Since $a \mid (b + 1)$, then $b + 1 = ay$ for some integer $y$. Solving for $b$ gives $b = ay - 1$. Equating the two expressions gives $ax = ay - 1$, or $ay - ax = 1$, which gives

\[
a(y - x) = 1.
\]

Since $a$ and $y - x$ are integers and $a \geq 2$, this is a contradiction. \[\square\]

9. Evaluate the proof of the following proposition.

**Proposition.** Let $n \in \mathbb{Z}$. If $3n - 8$ is odd, then $n$ is odd.

*Proof.* Assume that $n$ is odd. Then $n = 2k + 1$ for some integer $k$. Then

\[
3n - 8 = 3(2k + 1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k - 3) + 1.
\]

Since $3k - 3$ is an integer, $3n - 8$ is odd. \[\square\]

**Solution:** It appears that the person writing the proof tried to do a proof by contrapositive. However, what the proof really shows is that if $n$ is an odd integer, then $3n - 8$ is odd, the converse of the proposition. To prove the given proposition, we would use proof by contrapositive in which we would prove that if $n$ is an even integer, then $3n - 8$ is even.