In-Class Example (4) for Laurent Series
Math 128, Fall 2013
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Write the Laurent series expansion of \( f(z) = \frac{1}{(z-2)(z-1)} \)

The singularities of \( f(z) \) are \( z = 1 \) and \( z = 2 \). Therefore, there are three potential regions of interest, if we choose \( z_0 = 0 \). They are \( |z| < 1 \), \( 1 < |z| < 2 \), and \( 2 < |z| < \infty \).

(a) We can use partial fractions to write

\[
\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}.
\]

Let’s look at the Taylor series expansions of the above terms to hopefully gain some insight.

The Taylor series expansion of \( \frac{1}{1-z} \) about \( z = 0 \) is

\[
\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.
\]

Then,

\[
- \frac{1}{z-1} = \frac{1}{1-z},
\]

and

\[
\frac{1}{z-2} = -\frac{1}{2-z} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n = \sum_{n=0}^{\infty} \left( -\frac{1}{2^{n+1}} \right) z^n,
\]

which converges for \( 0 < \left| \frac{z}{2} \right| < 1 \implies |z| < 2 \). Therefore,

\[
\frac{1}{(z-2)(z-1)} = \sum_{n=0}^{\infty} \left( -\frac{1}{2^{n+1}} \right) z^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \left( 1 - \frac{1}{2^{n+1}} \right) z^n, \quad |z| < 1.
\]

(b) Suppose we want the Laurent expansion that converges to \( f(z) \) in the annulus \( 1 < |z| < 2 \).

In that domain, each point \( z \) satisfies \( |z| > 1 \implies \left| \frac{1}{z} \right| < 1 \) and \( \left| \frac{z}{2} \right| < 1 \). So, we need to rewrite
\[ f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z} \left( \frac{1}{1-\frac{1}{z}} - \frac{1}{1-\frac{1}{z}} \right) = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}}. \]

\[ \frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{z^n}, \]

which converges for \( \left| \frac{1}{z} \right| < 1 \implies 1 < |z| < \infty \). Then,

\[ = \sum_{n=0}^{\infty} \left( \frac{-1}{2n+1} \right) z^n - \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left( \frac{-1}{2n+1} \right) z^n + \sum_{n=1}^{\infty} \frac{1}{z^n+1}. \]

Therefore,

\[ f(z) = \sum_{n=0}^{\infty} \left( \frac{-1}{2n+1} \right) z^n + \sum_{n=1}^{\infty} \frac{1}{z^n+1}, \quad 1 < |z| < 2. \]

(c) Find the Laurent series for \( f(z) \) in the domain \( 2 < |z| < \infty \).

In this case, \( |z| > 2 \implies \left| \frac{2}{z} \right| < 1 \) and we need \( |z| > 1 \implies \left| \frac{1}{z} \right| < 1 \). So, rewrite \( f(z) \) again.

\[ f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z} \left( \frac{1}{z} - \frac{1}{1-\frac{1}{z}} \right) = \frac{1}{z} \left( \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n - \sum_{n=0}^{\infty} \frac{1}{z^n} \right) = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{(2^n-1)}{z^{n+1}}, \quad 2 < |z| < \infty. \]