Complex numbers may be defined as ordered pairs \((x, y)\) of real numbers \(x\) and \(y\). These ordered pairs are to be interpreted as points in the complex plane with rectangular coordinates \(x\) and \(y\).

Real numbers, \(x\), are displayed as points \((x, 0)\), so the \(x\)-axis is known as the real axis.

Complex numbers of the form \((0, y)\) correspond to points on the \(y\)-axis, and are called pure imaginary numbers when \(y \neq 0\). Therefore, the \(y\)-axis is the imaginary axis.

Complex numbers of the form \((x, y)\) are usually denoted by \(z\); i.e., \(z = (x, y)\).

Let \(z = (x, y)\). Then, define

\[
x = \text{Re} \, z \text{ (the real part of } z),
\]
\[
y = \text{Im} \, z \text{ (the imaginary part of } z).\]

Two complex numbers \(z_1 = (x_1, y_1)\) and \(z_2 = (x_2, y_2)\) are equal if, and only if, their real and imaginary parts are equal; i.e., \(z_1\) and \(z_2\) must correspond to the same point. Symbolically,

\[
z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2.
\]

The set of all complex numbers is denoted \(\mathbb{C}\). So, \(z \in \mathbb{C}\) means that \(z\) is a complex number.

## 1 Sums and Products – Section 1 of Brown and Churchill

Let \(z_1 = (x_1, y_1)\) and \(z_2 = (x_2, y_2)\). Since \(z_1\) and \(z_2\) represent two points in the complex plane, we can define addition of \(z_1\) and \(z_2\) in the same way that we define addition of any two points in the plane, i.e.,

\[
z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).
\]
Note that this translates into the standard addition of real numbers when $z_1$ and $z_2$ are real: $(x_1,0) + (x_2,0) = (x_1 + x_2,0)$, which is the same as $x_1 + x_2$.

The definition of multiplication of complex numbers is less natural, but it is as follows:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2).$$

(2)

Note that this translates into the standard multiplication of real numbers when $z_1$ and $z_2$ are real: $(x_1,0)(x_2,0) = (x_1 x_2,0)$, which is the same as $x_1 x_2$.

So, the complex number system is a natural extension of the real number system.

Any complex number $z = (x,y)$ can be written

$$z = (x,0) + (0,y),$$

Since

$$(0,1)(y,0) = (0 \cdot y - 1 \cdot 0, 1 \cdot y + 0 \cdot 1) = (0,y),$$

this means that any complex number can be written as

$$z = (x,0) + (0,1)(y,0).$$

If we think of a real number as either $x$ or $(x,0)$ and let $i$ denote the pure imaginary number $(0,1)$, then we can write

$$z = x + iy.$$

Using the standard convention that $z^2 = z \cdot z$, $z^3 = z^2 \cdot z$, etc., we have

$$i^2 = (0,1)(0,1) = (-1,0),$$

or

$$i^2 = -1.$$  

Since $(x,y) = x + iy$, definitions (1) and (2) become

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

(3)

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

(4)

Note: Equation (4) tells us that $z \cdot 0 = 0$ for any $z = x + iy$.

2 Basic Algebraic Properties – Section 2 of Brown and Churchill

Complex addition and multiplication satisfy many of the standard properties that real numbers satisfy.
• Commutative law: \( z_1 + z_2 = z_2 + z_1, \ z_1 z_2 = z_2 z_1. \)

Note: Since \( iy = yi \) by the commutative law, we can write either \( z = x + iy \) or \( z = x + yi \).

• Associative law: \( (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \ (z_1 z_2) z_3 = z_1 (z_2 z_3). \)

• Distributive law: \( z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3. \)

• Additive indentity (unique): \( z + 0 = z. \)

\[
[z = (x, y), 0 = (0, 0) \implies z + 0 = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y).]
\]

• Multiplicative identity (unique): \( z \cdot 1 = z. \)

• Additive inverse: Given \( z = (x, y) \), its additive inverse is \( -z = (-x, -y) \), satisfying \( z + (-z) = 0. \)

\(-z\) is the unique inverse. Why?

Let \( z = (x, y), w = (u, v) \). Then \( z + w = 0 \) only if

\[
(x, y) + (u, v) = (0, 0) \\
(x + u, y + v) = (0, 0).
\]

Since complex numbers are equal only if their real and imaginary parts are identical, this gives

\[
x + u = 0, \ y + v = 0 \implies u = -x, \ v = -y.
\]

• Multiplicative inverse: For any non-zero complex number \( z = (x, y) \) there is a number \( z^{-1} \) such that \( z z^{-1} = 1. \)

Determine \( z^{-1} \): Let \( z = (x, y), w = (u, v) \). We want \( (x, y)(u, v) = (1, 0) \). So,

\[
(x, y)(u, v) = (1, 0) \\
(xu - yv, yu + xv) = (1, 0) \\
\implies xu - yv = 1, \ yu + xv = 0.
\]

Solving for \( u \) and \( v \) gives

\[
u = \frac{x}{x^2 + y^2}, \ v = -\frac{y}{x^2 + y^2}.
\]

Therefore,

\[
z^{-1} = \left( \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} \right), \tag{5}
\]
3 Further Properties – Section 3 of Brown and Churchill

First, we discuss some consequences of the existence of multiplicative inverses.

1. If \( z_1z_2 = 0 \), then either \( z_1 = 0 \) or \( z_2 = 0 \).

2. Definition of quotients: If \( z_2 \neq 0 \), then \( \frac{z_1}{z_2} = z_1z_2^{-1} \), so

\[
\frac{z_1z_2^{-1}}{z_2} = (x_1, y_1) \left( \frac{x_2}{x_2^2 + y_2^2}, -\frac{y_2}{x_2^2 + y_2^2} \right) = \left( \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \right).
\]

We can also define subtraction:

\[
z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2).
\]

Subtraction and division – take two: Let \( z_1 = x_1 + iy_1 \), \( z_2 = x_2 + iy_2 \).

- \( z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \)
- \( \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \).

Calculating quotients made easy:

First, we note that

\[
\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1z_3^{-1} + z_2z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}.
\]

Using that fact, and the fact that \((x + iy)(x - iy) = x^2 + y^2\), we obtain:

\[
\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}.
\]

Example. Determine \( \frac{2 - 3i}{5 + 4i} \).

\[
\frac{2 - 3i}{5 + 4i} \cdot \frac{5 - 4i}{5 - 4i} = -\frac{2}{41} - \frac{23}{41}i.
\]
Exercise. Determine $\frac{5}{1+2i}$.

Solution. 
\[
\frac{5}{1+2i} \cdot \frac{1-2i}{1-2i} = 1-2i.
\]

Other fun facts:

1. Since $\frac{1}{z_2} = z_2^{-1}$,
\[
\frac{z_1}{z_2} = z_1 \left( \frac{1}{z_2} \right).
\]

2. If $z_1 \neq 0$ and $z_2 \neq 0$, then $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$. Why?
\[
(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \cdot 1 = 1.
\]

3. If $z_1 \neq 0$ and $z_2 \neq 0$, then
\[
\left( \frac{1}{z_1} \right) \left( \frac{1}{z_2} \right) = z_1^{-1}z_2^{-1} = \frac{1}{z_1z_2}.
\]

4. If $z_3 \neq 0$ and $z_4 \neq 0$, then
\[
\frac{z_1}{z_3} \cdot \frac{z_2}{z_4} = \frac{z_1z_2}{z_3z_4}.
\]

4 Vectors and Moduli – Section 4 of Brown and Churchill

It is natural to associate any nonzero complex number $z = x + iy$ with the vector from the origin to the point $(x, y)$ representing $z$ in the complex plane.

Example. $-3 + 2i$

\[
y
\]
\[
(-3, 2)
\]

\[
x
\]

\[
O
\]
When $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then
\[ z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \]
corresponding to the point $(x_1 + x_2, y_1 + y_2)$ and a vector with those coordinates as components (parallelogram rule for vector addition).

**Definition.** The modulus, or absolute value, of a complex number $z = x + iy$ is denoted by $|z|$ and defined
\[ |z| = \sqrt{x^2 + y^2}. \]  

So, the modulus of the complex number $z$ is the length of the vector representing $z$.

We can define the distance between two points: $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ as follows. Since $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$, then $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Geometrically, $|z_1 - z_2|$ is the length of the directed line segment from the point $(x_2, y_2)$ to the point $(x_1, y_1)$.

Circles in the complex plane: The complex numbers $z$ corresponding to the points lying on the circle with center $z_0$ and radius $R$ satisfy the equation $|z - z_0| = R$, and conversely.

**Example.** $|z + 2 - 3i| = 4$ represents the circle centered at $z_0 = (-2, 3)$ with radius $R = 4$.

**Useful Properties of Modulus**

1. Since $\text{Re} \, z = x$, $\text{Im} \, z = y$, we can write
\[ |z|^2 = (\text{Re} \, z)^2 + (\text{Im} \, z)^2. \]
   This implies
   \[ \text{Re} \, z \leq |\text{Re} \, z| \leq |z| \quad \text{and} \quad \text{Im} \, z \leq |\text{Im} \, z| \leq |z|. \]

2. **Triangle Inequality:**
\[ |z_1 + z_2| \leq |z_1| + |z_2|. \]  
   - Geometrically, this says that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.
   - Equality holds only if $0$, $z_1$, and $z_2$ are colinear.

3. $|z_1 + z_2| \geq ||z_1| - |z_2||$. Why?
\[ |z_1| = |(z_1 + z_2) + (-z_2)| \]
\[ \leq |z_1 + z_2| + |-z_2| \]
\[ = |z_1 + z_2| + |z_2| \]
\[ \implies |z_1 + z_2| \geq |z_1| - |z_2| \text{ if } |z_1| \geq |z_2|. \]
If $|z_1| < |z_2|$, then $|z_1 + z_2| \geq -(|z_1| - |z_2|)$.
Therefore, $|z_1 + z_2| \geq ||z_1| - |z_2||$. 


(4) Since $|z - z_2| = |z_2|$, we also obtain

(a) $|z_1 - z_2| \leq |z_1| + |z_2|$
(b) $|z_1 - z_2| \geq ||z_1| - |z_2||$

Example. Show that

$$\frac{1}{|z_1 - z_2|} \leq \frac{1}{||z_1| - |z_2||}$$

if $|z_1| \neq |z_2|$. Since $|z_1 - z_2| \geq ||z_1| - |z_2||$, multiplying both sides by $|z_1 - z_2|^{-1}||z_1| - |z_2||^{-1}$ gives the desired result.

5 Complex Conjugates – Section 5 of Brown and Churchill

Definition. The complex conjugate, or simply the conjugate, of a complex number $z = x + iy$, denoted by $\overline{z}$, is given by $\overline{z} = x - iy$.

The number $\overline{z}$ is represented by the point $(x, -y)$, which is the reflection about the real axis of the point $(x, y)$ representing $z$.

Some Properties of Complex Conjugates

1. $\overline{\overline{z}} = z$.
2. $||\overline{z}| = |z|$.
3. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2),$$

so $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
4. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.
5. $\overline{z_1 \overline{z_2}} = \overline{z_1} \overline{z_2}$.
6. $\frac{\overline{z_1}}{\overline{z_2}} = \frac{z_1}{z_2}$, provided $z_2 \neq 0$. 


(7) If \( z = x + iy \), then
\[
z + \overline{z} = (x + iy) + (x - iy) = 2x \quad \text{and} \quad z - \overline{z} = (x + iy) - (x - iy) = i(2y).
\]
In other words,
\[
\Re z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \Im z = \frac{z - \overline{z}}{2i}.
\]

(8) \( zz = |z|^2 \).

This suggests an easy method to determine \( \frac{z_1}{z_2} \) as:
\[
\frac{z_1}{z_2} = \frac{z_1 \overline{z}_2}{z_2 \overline{z}_2} = \frac{z_1 \overline{z}_2}{|z_2|^2}.
\]

**Example.** Determine \( \frac{2 - 3i}{1 + 2i} \).

\[
\frac{2 - 3i}{1 + 2i} = \frac{(2 - 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{(2 - 6) + i(-3 - 4)}{|1 - 2i|^2} = -\frac{4}{5} - \frac{7}{5}i.
\]

**Exercise.** Find \( \frac{2i}{3 - 4i} \).

**Solution.**
\[
\frac{2i}{3 - 4i} = \frac{2i(3 + 4i)}{|3 - 4i|^2} = -\frac{8}{13} + \frac{6}{13}i.
\]

We can use the properties of complex conjugates to derive addition properties of moduli.

**Example.** Show the following.

(a) \( |z_1z_2| = |z_1||z_2| \)
(b) \( \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0 \).

(a) Use the square of the modulus for simplicity.
\[
|z_1z_2|^2 = (z_1 z_2)(\overline{z}_1\overline{z}_2)
= (z_1 z_2)(\overline{z}_1\overline{z}_2)
= (z_1 \overline{z}_1)(z_2 \overline{z}_2)
= |z_1|^2|z_2|^2
= (|z_1||z_1|)^2.
\]
Taking square roots gives the desired result.

Note that this means that $|z|^2 = |z|^2$, $|z|^3 = |z|^3$, etc.

(b) Since $\frac{z_1}{z_2} = z_1z_2^{-1}$, the result easily follows from (a) and previous work.

Example. On the disc $|z| < 2$, bound

$$z^3 - 3z^2 + 2z + 8.$$ \[
|z^3 - 3z^2 + 2z + 8| \leq |z|^3 + 3|z|^2 + 2|z| + 8 \\
< (2)^3 + 3(2)^2 + 2(2) + 8 = 32.
\]

Exercise. If $|z| < 3$, bound the expression

$$z^2 + 8z - 12.$$ \[
|z^2 + 8z - 12| \leq |z|^2 + 8|z| + 12 \\
< (3)^2 + 8(3) + 12 = 45.
\]

6 Exponential Form – Section 6 of Brown and Churchill

6.1 Polar Form

Let $r$ and $\theta$ be polar coordinates of the point $(x, y)$ corresponding to a nonzero complex number $z = x + iy$.

Since $x = r \cos \theta$ and $y = r \sin \theta$, $z$ can be expressed in polar form as

$$z = r(\cos \theta + i \sin \theta). \quad (8)$$

If $z = 0$, then $\theta$ is undefined.

Since $r = \sqrt{x^2 + y^2}$, $r = |z|$.

The number $\theta$ represents the angle that $z$ makes with the positive $x$-axis when $z$ is interpreted as a vector from the origin. Note the following.
• \( \theta \) is determined by \( \tan \theta = \frac{y}{x} \).

• Each value of \( \theta \) is called an argument of \( z \), denoted \( \arg z \).

The principal value of \( \arg z \), denoted \( \text{Arg} z \), is the unique value \( \theta \) such that \(-\pi < \theta \leq \pi\). This means that

\[
\arg z = \text{Arg} z + 2n\pi \quad (n \in \mathbb{Z}).
\]

Note: When \( z \) is a negative real number, \( \text{Arg} z = \pi \).

**Example.** Find \( \text{Arg} z \) and \( \arg z \) for \( z = -1 + i \).

Since

\[
\tan \theta = \frac{1}{-1} = -1 \implies \theta = \frac{3\pi}{4}, \quad \text{and} \quad -\pi < \frac{3\pi}{4} \leq \pi,
\]

we have that \( \text{Arg} z = \frac{3\pi}{4} \). Based on the above, this tells us that \( \arg z = \frac{3\pi}{4} + 2n\pi, \; n \in \mathbb{Z} \).

**Exercise.** Find \( \text{Arg} z \) and \( \arg z \) for \( z = 1 - \sqrt{3}i \).

**Solution.** Since

\[
\tan \theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \implies \theta = -\frac{\pi}{3}, \quad \text{and} \quad -\pi < -\frac{\pi}{3} < \pi,
\]

we have that \( \text{Arg} z = -\frac{\pi}{3} \). Therefore, \( \arg z = -\frac{\pi}{3} + 2n\pi, \; n \in \mathbb{Z} \).

### 6.2 Exponential Form

We need one additional ingredient for exponential form: Euler’s formula,

\[
e^{i\theta} = \cos \theta + i \sin \theta.
\]  

Using this, we can write a complex number in exponential form,

\[
z = re^{i\theta}.
\]

Note: \( e^{i0} = 1, \; e^{i\pi} = -1, \; e^{i\frac{\pi}{2}} = i, \; e^{-i\frac{\pi}{2}} = -i \).

**Example.** Write \( z = -1 + i \) in exponential form.

\[
\begin{align*}
r &= \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\
\text{Arg} z &= \frac{3\pi}{4} \\
\implies z &= \sqrt{2}e^{i\frac{3\pi}{4}} \; \text{(only one possibility), or} \\
z &= \sqrt{2}e^{i\left(\frac{3\pi}{4} + 2n\pi\right)}, \; n \in \mathbb{Z}.
\end{align*}
\]
Exercise. Write $z = 1 - \sqrt{3}i$ in exponential form.

Solution. $z = 2e^{i(-\frac{\pi}{3} + 2n\pi)}$, $n \in \mathbb{Z}$.

Notes:

- The equation $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$, is a parametric representation of the circle $|z| = R$.
- The equation $z = z_0 + Re^{i\theta}$, $0 \leq \theta \leq 2\pi$, is a parametric representation of the circle $|z - z_0| = R$.

7 Products and Powers in Exponential Form – Section 7 of Brown and Churchill

7.1 Multiplication and Division of Complex Numbers in Exponential Form

First, we must understand how to multiply and divide $e^{i\theta}$.

$$e^{i\theta_1}e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$
$$= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$
$$= e^{i(\theta_1 + \theta_2)}.$$  

Similarly, we see

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{(\cos \theta_1 + i \sin \theta_1)}{(\cos \theta_2 + i \sin \theta_2)}$$
$$= \frac{(\cos \theta_1 + i \sin \theta_1)}{(\cos \theta_2 + i \sin \theta_2)} \cdot \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)}$$
$$= \frac{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2}$$
$$= \cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$
$$= e^{i(\theta_1 - \theta_2)}.$$
So, if \( z_1 = r_1 e^{i\theta_1} \) and \( z_2 = r_2 e^{i\theta_2} \), then we have the following:

\[
\begin{align*}
  z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\
  \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \quad \text{(provided } z_2 \neq 0) \\
  z^{-1} &= \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{i(0 - \theta)} = \frac{1}{r} e^{-i\theta}.
\end{align*}
\]

### 7.2 Powers of \( z = r e^{i\theta} \)

By repeatedly applying the rules for products and quotients above, we see that for any \( n \in \mathbb{Z} \),

\[ z^n = r^n e^{i(n\theta)}. \]

**Example.** Determine \((-1 + i)^{10}\).

From a previous example, we know that \(-1 + i = \sqrt{2} e^{i(\frac{3\pi}{4})}\). Therefore,

\[
\begin{align*}
(-1 + i)^{10} &= (\sqrt{2} e^{i(\frac{3\pi}{4})})^{10} \\
&= (\sqrt{2})^{10} e^{i(10 \cdot \frac{3\pi}{4})} \\
&= 2^5 e^{i\frac{15\pi}{2}} \\
&= 2^5 e^{i(-\frac{\pi}{2} + 8\pi)} \\
&= 2^5 e^{-i\frac{\pi}{2}} (e^{i\pi})^8 \\
&= 32(-i)(-1)^8 \\
&= -32i.
\end{align*}
\]

**Exercise.** Determine \((1 + \sqrt{3}i)^5\).

**Solution.** Verify that \(1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}\). Then,

\[
\begin{align*}
(1 + \sqrt{3}i)^5 &= 2^5 e^{i5(\frac{\pi}{3})} \\
&= 32 e^{i\frac{5\pi}{3}} \\
&= 32 e^{-i\frac{\pi}{3}} \\
&= 32 \left( \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right) \\
&= 32 \left( \frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \\
&= 16 - 16\sqrt{3}i.
\end{align*}
\]
8 Arguments of Products and Quotients – Section 8 of Brown and Churchill

8.1 Arguments of Products

If \( z_1 = r_1e^{i\theta_1} \) and \( z_2 = r_2e^{i\theta_2} \), we saw that

\[
z_1z_2 = (r_1r_2)e^{i(\theta_1 + \theta_2)} \tag{10}
\]

\[\implies \arg(z_1z_2) = \arg z_1 + \arg z_2. \tag{11}\]

Why?

- Let \( \theta_1 \) and \( \theta_2 \) denote any values of \( \arg z_1 \) and \( \arg z_2 \), respectively.
- Then, (10) tells us that \( \theta_1 + \theta_2 \) is a value of \( \arg(z_1z_2) \).
- Now, suppose values of \( \arg(z_1z_2) \) and \( \arg z_1 \) are specified. Then

\[
\arg(z_1z_2) = (\theta_1 + \theta_2) + 2n\pi, \quad n \in \mathbb{Z}
\]

\[
\arg z_1 = \theta_1 + 2n_1\pi, \quad n_1 \in \mathbb{Z}.
\]

Since \( (\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + (\theta_2 + 2(n - n_1)\pi) \), (11) is satisfied when

\[
\arg z_1 = \theta_2 + 2(n - n_1)\pi
\]

is chosen. Similarly, we can do this if values of \( \arg(z_1z_2) \) and \( \arg z_2 \) are specified.

NOTE: (11) is not always true if \( \arg \) is replaced by \( \text{Arg} \).

Examples.

(1) Let \( z_1 = -i \), \( z_2 = 1 \). Does \( \text{Arg} (z_1z_2) = \text{Arg} z_1 + \text{Arg} z_2 \)?

Since \( z_1 = e^{-i\frac{\pi}{2}} \), \( z_2 = e^{i0} \), \( \text{Arg} z_1 = -\frac{\pi}{2} \) and \( \text{Arg} z_2 = 0 \).

\[
z_1z_2 = -i, \quad \text{so Arg} (z_1z_2) = -\frac{\pi}{2}.
\]

In this case, we have

\[
\text{Arg} z_1 + \text{Arg} z_2 = -\frac{\pi}{2} + 0 = -\frac{\pi}{2} = \text{Arg} (z_1z_2).
\]
(2) Let \( z_1 = -1 + i \) and \( z_2 = i \). Does \( \text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2 \)?

In this case,
\[
\text{Arg} z_1 = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4} \quad \text{and} \quad \text{Arg} z_2 = \frac{\pi}{2}.
\]

Since \( z_1 z_2 = -1 - i \),
\[
\text{Arg}(z_1 z_2) = \tan^{-1}\left(-\frac{1}{-1}\right) = -\frac{3\pi}{4}.
\]

But,
\[
\text{Arg} z_1 + \text{Arg} z_2 = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4} \neq \text{Arg}(z_1 z_2).
\]

### 8.2 Arguments of Quotients

Let \( z_1 = r_1 e^{i\theta_1} \) and \( z_2 = r_2 e^{i\theta_2} \). Then, we have that
\[
\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 z_2^{-1}) = \arg z_1 + \arg z_2^{-1}.
\]

Since \( z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2} \), \( \arg(z_2^{-1}) = -\theta_2 = -\arg z_2 \). Therefore,
\[
\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.
\]

**Example.** Find \( \text{Arg} z \) if
\[
z = \frac{2}{1 - i}.
\]

Let \( z_1 = 2 \) and \( z_2 = 1 - i \). Then, since \( z = \frac{z_1}{z_2} \), we have that \( \arg z = \arg z_1 - \arg z_2 \).

\[
\begin{align*}
z_1 &= 2e^{i0} \implies \arg z_1 = 0 \\
z_2 &= 1 - i = \sqrt{2}e^{-i\frac{\pi}{4}} \implies \arg z_2 = -\frac{\pi}{4}.
\end{align*}
\]

So,
\[
\arg z = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}.
\]

Since \(-\pi < \frac{\pi}{4} \leq \pi\), \( \text{Arg} z = \frac{\pi}{4} \).
Exercise. Find $\text{Arg } z$ when

$$z = \frac{2}{1 - \sqrt{3}i}.$$ 

Solution. Let $z_1 = 2$ and $z_2 = 1 - \sqrt{3}i$. Then, since $z = \frac{z_1}{z_2}$, we have that $\text{arg } z = \text{arg } z_1 - \text{arg } z_2$.

$$z_1 = -2 = 2e^{i\pi} \implies \text{arg } z_1 = \pi$$

$$z_2 = 1 - \sqrt{3}i = 2e^{-i\frac{\pi}{3}} \implies \text{arg } z_2 = -\frac{\pi}{3}.$$ 

So,

$$\text{arg } z = \pi - \left(-\frac{\pi}{3}\right) = \frac{4\pi}{3}.$$ 

Since $\frac{4\pi}{3} > \pi$, we need to subtract $2\pi$ in order to obtain $\text{Arg } z$. If we do this, we see that

$$\text{Arg } z = -\frac{2\pi}{3}.$$ 

9 Roots of Complex Numbers – Section 9 of Brown and Churchill

9.1 Equality of Two Complex Numbers in Exponential Notation

Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. For $z_1$ and $z_2$ to be equal:

- They must be points on the same circle, so $r_1 = r_2$.
- They must be the same point on the circle. But, as we know, if an angle $\theta$ is increased (or decreased) by $2\pi$, we get back to the original point. So, we need $\theta_1 = \theta_2 + 2k\pi$, $k \in \mathbb{Z}$.

Therefore, $z_1 = z_2$ if and only if

$$r_1 = r_2$$

and $\theta_1 = \theta_2 + 2k\pi$, $k \in \mathbb{Z}$.

9.2 Finding $n$th Roots of a Complex Number

Let $z_0 = r_0e^{i\theta_0}$. 

If we want to find the $n$th root of $z_0$ (where $n = 2, 3, \ldots$), then we seek a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$  

$$\implies r^n = r_0 \text{ and } n\theta = \theta_0 + 2k\pi, \ k \in \mathbb{Z}$$  

$$\implies r = (r_0)^{\frac{1}{n}} \text{ and } \theta = \frac{\theta_0 + 2k\pi}{n}, \ k \in \mathbb{Z}.$$  

In other words,

$$z = (r_0)^{\frac{1}{n}} e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}.$$  

This means that:

- All $n$th roots of $z_0$ lie on the circle $|z| = (r_0)^{\frac{1}{n}}$ about the origin and are equally spaced every $\frac{2\pi}{n}$ radians, starting with the argument $\frac{\theta_0}{n}$.
- All distinct roots are thus obtained when $k = 0, 1, 2, \ldots, n - 1$.

Let $c_k$ denote the distinct roots of $z_0$. $c_0$ is known as the **principal root**. Then

$$c_k = (r_0)^{\frac{1}{n}} e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}, \ k = 0, 1, 2, \ldots, n - 1.$$  

To find the root in practice, do the following.

- Write $z_0 = r_0 e^{i(\theta_0 + 2k\pi)}, \ k \in \mathbb{Z}$.
- Then

$$\left(z_0\right)^{\frac{1}{n}} = \left[r_0 e^{i(\theta_0 + 2k\pi)}\right]^{\frac{1}{n}}$$  

$$= (r_0)^{\frac{1}{n}} e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}, \ k \in \mathbb{Z}.$$  

- So,

$$c_k = (r_0)^{\frac{1}{n}} e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}, \ k = 0, 1, 2, \ldots, n - 1.$$  

### 10 Examples – Section 10 of Brown and Churchill

(1) Find all values of $(-8i)^{\frac{1}{3}}$ in rectangular coordinates.

$$z_0 = -8i \implies r_0 = 8, \ \theta_0 = -\frac{\pi}{2}$$  

So, $-8i = 8e^{i\left(-\frac{\pi}{2} + 2k\pi\right)}, \ k \in \mathbb{Z}$

$$\implies c_k = 8^{\frac{1}{3}} e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)}, \ k = 0, 1, 2$$  

$$= 2e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)}, \ k = 0, 1, 2.$$
\[ k = 0: \quad c_0 = 2e^{i \left( -\frac{\pi}{6} \right)} = 2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = \sqrt{3} - i. \]

\[ k = 1: \quad c_1 = 2e^{i \left( -\frac{\pi}{6} + \frac{2\pi}{3} \right)} = 2e^{i \frac{\pi}{2}} = 2i. \]

\[ k = 2: \quad c_2 = 2e^{i \left( -\frac{\pi}{6} + \pi \right)} = 2e^{i \frac{7\pi}{6}} = 2 \left( \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) = -\sqrt{3} - i. \]

So, we see that
\[ (-8i)^{\frac{1}{4}} = \sqrt{3} - i, \quad 2i, \quad -\sqrt{3} - i. \]

2) Find all roots of \((-8 - 8\sqrt{3}i)^{\frac{1}{4}}\) in rectangular coordinates.

\[ z_0 = -8 - 8\sqrt{3}i \implies r_0 = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = 16 \]
\[ \theta_0 = \tan^{-1} \left( \frac{-8\sqrt{3}}{-8} \right) = -\frac{2\pi}{3}. \]

So, we have
\[ -8 - 8\sqrt{3}i = 16e^{i \left( -\frac{2\pi}{3} + 2k\pi \right)}, \quad k \in \mathbb{Z}. \]

Therefore, the roots satisfy
\[ c_k = \left( -8 - 8\sqrt{3}i \right)^{\frac{1}{4}} = 16^{\frac{1}{4}}e^{i \left( -\frac{2\pi}{3} + 2k\pi \right)}, \quad k = 0, 1, 2, 3 \]
\[ = 2e^{i \left( -\frac{\pi}{6} + k\pi \right)}, \quad k = 0, 1, 2, 3. \]

\[ k = 0: \quad c_0 = 2e^{i \left( -\frac{\pi}{6} \right)} = \sqrt{3} - i \]
\[ k = 1: \quad c_1 = 2e^{i \left( -\frac{\pi}{6} + \frac{\pi}{3} \right)} = 1 + \sqrt{3}i \]
\[ k = 2: \quad c_2 = 2e^{i \left( -\frac{\pi}{6} + \pi \right)} = -\sqrt{3} + i \]
\[ k = 3: \quad c_3 = 2e^{i \left( -\frac{\pi}{6} + \frac{2\pi}{3} \right)} = -1 - \sqrt{3}i \]

So,
\[ (-8 - 8\sqrt{3}i)^{\frac{1}{4}} = \sqrt{3} - i, \quad 1 + \sqrt{3}i, \quad -\sqrt{3} + i, \quad -1 - \sqrt{3}i. \]

**Exercise.** Find all roots of \((-5)^{\frac{1}{4}}\) in rectangular coordinates.

**Solution.** Let \(z_0 = -5\). Then, \(r_0 = 5\) and \(\theta_0 = \pi\). So, we have
\[ -5 = 5e^{i (\pi + 2k\pi)}, \quad k \in \mathbb{Z}. \]

Therefore, the roots satisfy
\[ c_k = (-5)^{\frac{1}{4}} = 5^{\frac{1}{4}}e^{i \left( \pi + 2k\pi \right)}, \quad k = 0, 1, 2, 3 \]
\[ = 2e^{i \left( -\frac{\pi}{6} + k\pi \right)}, \quad k = 0, 1, 2, 3. \]
\[ k = 0: \quad c_0 = (5)^{\frac{1}{4}} e^{i \left( \frac{\pi}{4} \right)} = \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} + i \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} \]

\[ k = 1: \quad c_1 = (5)^{\frac{1}{4}} e^{i \left( \frac{\pi}{4} + \frac{\pi}{2} \right)} = (5)^{\frac{1}{4}} e^{i \left( \frac{3\pi}{4} \right)} = -\frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} + i \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} \]

\[ k = 2: \quad c_2 = (5)^{\frac{1}{4}} e^{i \left( \frac{\pi}{4} + \pi \right)} = (5)^{\frac{1}{4}} e^{i \left( \frac{7\pi}{4} \right)} = \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} - i \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} \]

\[ k = 3: \quad c_3 = (5)^{\frac{1}{4}} e^{i \left( \frac{\pi}{4} + \frac{3\pi}{2} \right)} = (5)^{\frac{1}{4}} e^{i \left( \frac{3\pi}{4} \right)} = -\frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} - i \frac{(5)^{\frac{1}{4}} \sqrt{2}}{2} \]

### 11 Regions in the Complex Plane – Section 11 of Brown and Churchill

This section contains some preliminary information from analysis that we will use later in the course.

**Definitions.**

- An **\( \epsilon \)-neighborhood** (or just **neighborhood**) of a point \( z_0 \in \mathbb{C} \) is the set of all points \( z \in \mathbb{C} \) such that \( |z - z_0| < \epsilon \).

- A **deleted neighborhood** of a point \( z_0 \in \mathbb{C} \) is the set of all points \( z \in \mathbb{C} \) such that \( 0 < |z - z_0| < \epsilon \) (i.e., it is the set of all points in an \( \epsilon \)-neighborhood of \( z_0 \) except \( z_0 \), itself).

**Definitions.** Let \( S \) be a set in the complex plane.

1. \( z_0 \) is an **interior point of** \( S \) whenever there exists a neighborhood of \( z_0 \) that contains only points in \( S \).

2. \( z_0 \) is an **exterior point of** \( S \) whenever there exists a neighborhood of \( z_0 \) that contains no points in \( S \).

3. \( z_0 \) is a **boundary point** of \( S \) if every neighborhood of \( z_0 \) contains at least one point in \( S \) and at least one point not in \( S \). The set of all boundary points of \( S \) is the **boundary of** \( S \).

**Example.** The circle \( |z| = 1 \) is the boundary of both \( |z| < 1 \) and \( |z| \leq 1 \).

**Definitions.** Let \( S \) be a set in the complex plane.
(1) $S$ is **open** if it contains none of its boundary points.

(2) $S$ is **closed** if it contains all of its boundary points.

(3) The **closure** of $S$ is the closed set consisting of all points in $S$ together with the boundary of $S$.

**Examples:**

(a) $|z - 2| < 1$ is an open set.

(b) $|z - 2| = 1$ is the boundary of the set in (a). It is also a closed set.

(c) $|z - 2| \leq 1$ is the closure of the set in (a).

Sets can be neither open nor closed.

**Example.** While the set $0 < |z - 2| \leq 1$ is neither open nor closed, $0 < |z - 2| < 1$ is open.

**Definitions.**

(1) An open set $S$ is **connected** if each pair of points $z_1, z_2 \in S$ can be connected by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in $S$.

(2) A nonempty, open set that is connected is a **domain**.

**Example.**
**Definition.** A set $S$ is **bounded** if there exists $R \in \mathbb{R}$ such that every point of $S$ lies in $|z| = R$. Otherwise, $S$ is **unbounded**.