Consider a complex-valued function $w$ of a real variable $t$:

$$w(t) = u(t) + iv(t).$$

The derivative of $w$, $w'(t) = \frac{d}{dt}w(t)$, at a point $t$ is given by

$$w'(t) = u'(t) + iv'(t),$$

provided $u'$ and $v'$ exist at $t$.

Let $z_0 = x_0 + iy_0$ (where $z_0$ is constant). Then

$$\frac{d}{dt}[z_0w(t)] = [(x_0 + iy_0)(u(t) + iv(t))]'$$

$$= [(x_0u(t) - y_0v(t)) + i(y_0u(t) + x_0v(t))]'$$

$$= (x_0u(t) - y_0v(t))' + i(y_0u(t) + x_0v(t))'$$

$$= (x_0u'(t) - y_0v'(t)) + i(y_0u'(t) + x_0v'(t))$$

$$= (x_0 + iy_0)(u'(t) + iv'(t))$$

$$= z_0w'(t).$$

So, $\frac{d}{dt}[z_0w(t)] = z_0w'(t)$.

**Example.** Find $\frac{d}{dt}(e^{z_0 t})$, where $z_0 = x_0 + iy_0$.

$$e^{z_0 t} = e^{(x_0 + iy_0)t}$$

$$= e^{x_0t}e^{iy_0t}$$

$$= e^{x_0t}(\cos(y_0 t) + i \sin(y_0 t)).$$
Differentiating,
\[
\frac{d}{dt}(e^{zt}) = \frac{d}{dt}(e^{x_0 t} \cos(y_0 t)) + i \frac{d}{dt}(e^{x_0 t} \sin(y_0 t))
\]
\[
= (x_0e^{x_0 t} \cos(y_0 t) - y_0e^{x_0 t} \sin(y_0 t)) + i(x_0e^{x_0 t} \sin(y_0 t) + y_0e^{x_0 t} \cos(y_0 t))
\]
\[
= (x_0 + iy_0)(e^{x_0 t} \cos(y_0 t) + ie^{x_0 t} \sin(y_0 t))
\]
\[
= z_0 e^{z_0 t}.
\]

Many rules from Calculus apply, but not all. For example, the Mean Value Theorem is not valid for complex-valued functions.

## 2 Definite Integrals of Functions $w(t) – Section 38$ of Brown and Churchill

Let $w(t) = u(t) + iv(t)$, $t \in \mathbb{R}$, where $u$ and $v$ are real-valued. Then,
\[
\int_a^b w(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt,
\]
provided the integrals on the right exist.

This implies that
\[
\text{Re} \int_a^b w(t) \, dt = \int_a^b \text{Re}[w(t)] \, dt, \quad \text{and}
\]
\[
\text{Im} \int_a^b w(t) \, dt = \int_a^b \text{Im}[w(t)] \, dt.
\]

**Example.** Evaluate $\int_0^1 (t - i)^2 \, dt$.
\[
\int_0^1 (t - i)^2 \, dt = \int_0^1 ((t^2 - 1) - 2ti) \, dt
\]
\[
= \int_0^1 (t^2 - 1) \, dt + i \int_0^1 (-2t) \, dt
\]
\[
= \left. \frac{1}{3}t^3 \right|_0^1 - t \big|_0^1 + i \left. (-t^2) \right|_0^1
\]
\[
= \frac{1}{3} - 1 + i(-1)
\]
\[
= \frac{2}{3} - i.
\]
Properties of definite integrals of $w(t)$:

1. If $c \in [a, b]$, then $\int_a^b w(t) \, dt = \int_a^c w(t) \, dt + \int_c^b w(t) \, dt$

2. If $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on $[a, b]$ and $W'(t) = w(t)$ for $t \in [a, b]$, then $U'(t) = u(t)$ and $V'(t) = v(t)$, and

$$\int_a^b w(t) \, dt = U(t)|_a^b + i V(t)|_a^b$$

$$= U(b) + iV(b) - (U(a) + iV(a))$$

$$= W(b) - W(a)$$

$$= W(t)|_a^b.$$

Example. Evaluate $\int_0^\pi e^{it} \, dt$.

$$\int_0^\pi e^{it} \, dt = \left. \frac{1}{i} e^{it} \right|_0^\pi$$

(since $\frac{d}{dt} \left( \frac{1}{i} e^{it} \right) = \frac{1}{i} \frac{d}{dt} (e^{it}) = \frac{i}{i} e^{it} = e^{it}$)

$$= \frac{1}{i} (e^{i\pi} - e^{i(0)})$$

$$= \frac{1}{i} (-1 - 1)$$

$$= \frac{-2}{i}$$

$$= 2i.$$

3 Contours – Section 39 of Brown and Churchill

Integrals of complex-valued functions of a complex variable are defined on curves in the complex plane.

A set of points $z = (x, y)$ in the complex plane is said to be an arc if

$$x = x(t), \ y = y(t), \ a \leq t \leq b,$$

where $x$ and $y$ are continuous functions of the real parameter $t$.

- This establishes a continuous mapping of the interval $a \leq t \leq b$ into the $xy$- or $z$-plane.
- Image points are ordered according to increasing values of $t.$
It is convenient to describe the points of an arc $C$ by
\[ z = z(t), \quad (a \leq t \leq b), \tag{2} \]
where $z(t) = x(t) + iy(t)$.

**Definitions.**
- The arc $C$ is a **simple arc**, or Jordan arc, if it does not cross itself.
- If the arc $C$ is simple, except that $z(b) = z(a)$, then $C$ is a **simple closed curve**, or a Jordan curve.
- A simple closed curve is **positively oriented** when it is in the counterclockwise direction.

**Examples.**

1. \[ z = \begin{cases} 1 + it, & 0 \leq t \leq 2 \\ t - 1 + 2i, & 2 \leq t \leq 4 \end{cases} \]

As is evident from the graph, this is a simple arc.

2. \[ z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \]
   is a simple closed curve oriented in the counterclockwise direction (i.e., positively oriented).

3. \[ z = 2 + e^{-i\theta}, \quad 0 \leq \theta \leq 2\pi \]
   is the circle centered at $(2, 0)$ and oriented in the clockwise direction; it is a simple closed curve.

**Exercise.** Give a parameterization for the curve defined by the line from $0$ to $1 + i$, followed by the line from $1 + i$ to $1$.

**Solution.** First, sketch the curve.
The first line is simply \( y = x \), so the parameterization of this line is done by letting \( x = t \). Then, since \( y = x, y = t \) and \( z = t + it \) or \( z = (1 + i)t \), \( 0 \leq t \leq 1 \). The second line is the line \( x = 1 \), so the parameterization is obtained by letting \( x = 1 \), \( y = t \), or \( z = 1 + it \), \( 1 \leq t \leq 0 \), but we want \( t \) to be increasing from 1 to 2 (so \( t \) for the entire curve ranges from 0 to 2), so we write \( z = 1 + (2 - t)i \), \( 1 \leq t \leq 2 \). Thus, the parametrized curve is

\[
z = \begin{cases} (1 + i)t, & 0 \leq t \leq 1, \\ 1 + (2 - t)i, & 1 \leq t \leq 2. \end{cases}
\]

The parametric representation used for any given arc is not unique. For example you can change the interval over which the parameter ranges to any given interval. Suppose \( t = \phi(\tau) \) (\( \alpha \leq \tau \leq \beta \)), where \( \phi \) is a real-valued function mapping \( \alpha \leq \tau \leq \beta \) onto \( a \leq t \leq b \) in representation (1). Assume that \( \phi \) is continuous with continuous derivative and \( \phi'(\tau) > 0 \). Then, (1) is transformed into

\[
z = Z(\tau), \ \alpha \leq \tau \leq \beta, \tag{3}
\]

where \( Z(\tau) = z[\phi(\tau)] \).

If \( x' \) and \( y' \) are continuous on \( [a, b] \), then the arc \( z(t) = x(t) + iy(t) \) is a differentiable arc \( (z'(t) = x'(t) + iy'(t)) \), and

\[
|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}
\]

is integrable on \( [a, b] \). From Calculus, the length of \( C \) is

\[
L = \int_{a}^{b} |z'(t)| \, dt.
\]

The value of \( L \) does not depend on the parameterization, so

\[
L = \int_{\alpha}^{\beta} |z'[\phi(\tau)]| \phi'(\tau) \, d\tau.
\]

So, if (3) is used for \( C \), then

\[
Z'(\tau) = z'[\phi(\tau)]\phi'(\tau)
\]

\[
\implies L = \int_{\alpha}^{\beta} |Z'(\tau)| \, d\tau.
\]
Definitions.

(1) A contour, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs.

(2) A smooth arc is one whose unit tangent vector is well-defined with angle of inclination that changes smoothly as $t$ varies over $[a, b]$.

(3) A simple closed contour is a contour that does not intersect itself, except that the initial and final values of $z(t)$ are identical.

4 Contour Integrals – Section 40 of Brown and Churchill

The integral of a complex-valued function $f(z)$, $z \in \mathbb{C}$, is defined in terms of the values $f(z)$ takes along a given contour $C$ extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane (i.e., it is a line integral).

Notation:

(a) $\int_C f(z) \, dz$

(b) $\int_{z_1}^{z_2} f(z) \, dz$

(b) is used when the value of the integral is path-independent (i.e., independent of the contour).

Suppose $z = z(t)$ ($a \leq t \leq b$) represents a contour $C$, extending from a point $z_1 = z(a)$ to $z_2 = z(b)$.

→ Assume that $f[z(t)]$ is piecewise continuous on $[a, b]$ and refer to $f(z)$ as piecewise continuous on $C$.

→ Define the contour integral

$$\int_C f(z) \, dz = \int_a^b f[z(t)]z'(t) \, dt.$$ 

Note: $z'(t)$ is piecewise continuous on $[a, b]$ since $C$ is a contour.

The value of a contour integral is invariant under a change in the representation of the contour when the change is of the form discussed in Section 39 (Contours).
Properties of Contour Integrals

(1) \( \int_C z_0 f(z) \, dz = z_0 \int_C f(z) \, dz \)

(2) \( \int_C (f(z) + g(z)) \, dz = \int_C f(z) \, dz + \int_C g(z) \, dz \)

(3) Associated with contour \( C \) is contour \( -C \) consisting of the same set of points but with the contour extending from \( z_2 \) to \( z_1 \). For example, see the picture below.

\[ \implies -C \text{ has parametric representation} \]
\[ z = z(-t), \quad -b \leq t \leq -a. \]

So,
\[ \int_{-C} f(z) \, dz = -\int_C f(z) \, dz. \]

Why?
\[ \int_{-C} f(z) \, dz = \int_{-b}^{-a} f[z(-t)] \left[ \frac{d}{dt} z(-t) \right] \, dt \]
\[ = -\int_{-b}^{-a} f[z(-t)] z'(-t) \, dt \]
\[ = -\int_{a}^{b} f[z(\tau)] z'('\tau) \, d\tau \]
\[ = -\int_C f(z) \, dz. \]

(4) We say \( C = C_1 + C_2 \) if \( C \) consists of a contour \( C_1 \) from \( z_1 \) to \( z_2 \) followed by a contour \( C_2 \) from \( z_2 \) to \( z_3 \). Then, there is a value \( c \in (a, b) \) such that \( z(c) = z_2 \). So,
\[ C_1 : z = z(t) \quad (a \leq t \leq c) \]
\[ C_2 : z = z(t) \quad (c \leq t \leq b). \]

and
\[ \int_a^b f[z(t)] z'(t) \, dt = \int_a^c f[z(t)] z'(t) \, dt + \int_c^b f[z(t)] z'(t) \, dt \]
\[ \implies \int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz. \]
5 Some Examples – Section 41 of Brown and Churchill

(1) Evaluate $\int_C z^2 \, dz$ where $C$ is the line segment from 0 to $2 + i$.

A parametric equation for $C$ is

$$z(t) = (2 + i)t, \; 0 \leq t \leq 1.$$  

Then,

$$\int_C z^2 \, dz = \int_0^1 [(2 + i)t^2][(2 + i)t]' \, dt$$

$$= \int_0^1 [(4 - 1 + i(4))t^2](2 + i) \, dt$$

$$= \int_0^1 [(3 + 4i)t^2](2 + i) \, dt$$

$$= \int_0^1 (3 + 4i)(2 + i) \, dt$$

$$= (3 + 4i)(2 + i) \int_0^1 t^2 \, dt$$

$$= \frac{2}{3} + \frac{11i}{3}.$$  

(2) Evaluate $\int_C z^2 \, dz$ where $C$ is the union of the line segment from $z = 0$ to $z = 2$ and then from $z = 2$ to $z = 2 + i$.

![Diagram of C1 and C2]

$C_1: \; z(t) = t, \; 0 \leq t \leq 2$

$C_2: \; z(t) = 2 + (t - 2)i, \; 2 \leq t \leq 3$
So,
\[
\int_C z^2 \, dz = \int_{C_1} z^2 \, dz + \int_{C_2} z^2 \, dz
\]
\[
= \int_0^2 t^2 \cdot 1 \, dt + \int_2^3 \left[2 + (t - 2)i\right]^2 i \, dt
\]
\[
= \frac{8}{3} + i \int_2^3 (4 - (t - 2)^2 + 4(t - 2)i) \, dt
\]
\[
= \frac{8}{3} + i \left[\frac{11}{3} + 2i\right]
\]
\[
= \left[\frac{2}{3} + \frac{11}{3}i\right].
\]

Is this true in general? We will see.

(3) Evaluate \(\int_{C_1} \overline{z} \, dz\), where \(C_1\) is the upper half of the circle \(|z| = 1\) from \(z = -1\) to \(z = 1\).

A parameterization for \(-C_1\) is \(z(\theta) = e^{i\theta}, \, 0 \leq \theta \leq \pi\). So,
\[
\int_{C_1} \overline{z} \, dz = -\int_{-C_1} \overline{z} \, dz
\]
\[
= -\int_0^\pi e^{-i\theta} \cdot ie^{i\theta} \, d\theta
\]
\[
= -\pi i.
\]

(4) Re-do (3) on the lower half of the unit circle.

A parameterization for \(C_2\) is \(z = e^{i\theta}, \pi \leq \theta \leq 2\pi\).
\[
\int_{C_2} \overline{z} \, dz = \int_\pi^{2\pi} e^{-i\theta} \cdot ie^{i\theta} \, d\theta
\]
\[
= \pi i.
\]
Note: If $C$ is the unit circle, then $C = -C_1 + C_2$, where $C_1$ is from Example (3) and $C_2$ is from this example. So, 

$$\int_C z \, dz = -(-\pi i) + \pi i = 2\pi i.$$ 

(5) Evaluate $\int_C (z - z_0)^n \, dz$, where $C$ is the circle $|z - z_0| = r$.

So, $z = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$ is a parameterization for $C$.

$$\int_C (z - z_0)^n \, dz = \int_0^{2\pi} (z_0 + re^{i\theta} - z_0) \cdot ire^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} r^n e^{i(n+1)\theta} \, d\theta$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta$$

$$= ir^{n+1} \left[ e^{i(n+1)\theta} \right]_0^{2\pi} \left( \text{if } n \neq -1 \right)$$

$$= 0.$$

If $n = -1$, then

$$\int_C (z - z_0)^n \, dz = i \int_0^{2\pi} d\theta = 2\pi i.$$

Exercise: Evaluate $\int_C (z - 1) \, dz$, where $C$ is the bottom half of the circle of radius 1 centered at $(1, 0)$ traversed counterclockwise.

Solution:

A parameterization of $C$ is $z(\theta) = 1 + e^{i\theta}$, $\pi \leq \theta \leq 2\pi$. Then,

$$\int_C (z - 1) \, dz = 0.$$
6 Upper Bounds for Moduli of Contour Integrals – Section 43 of Brown and Churchill

Lemma 1. If \( w(t) \) is a piecewise continuous complex-valued function defined on an interval \( a \leq t \leq b \), then

\[
\left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt.
\]

Proof. If \( \int_a^b w(t) \, dt = 0 \), the result is clearly true. So, suppose \( \int_a^b w(t) \, dt \neq 0 \). Then, since \( \int_a^b w(t) \, dt \) is a complex number, we may write it in exponential notation:

\[
\int_a^b w(t) \, dt = r_0 e^{i\theta_0}.
\]

So,

\[
 r_0 = e^{-i\theta_0} \int_a^b w(t) \, dt = \int_a^b e^{-i\theta_0} w(t) \, dt.
\]

Note that \( r_0 \) is a real number, so \( \int_a^b e^{-i\theta_0} w(t) \, dt \) must be a real number, as well. Since \( \text{Re} \, x = x, \, x \in \mathbb{R} \), we can write

\[
 r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t) \, dt
\]

\[
 = \int_a^b \text{Re} \left[ \int_a^b e^{-i\theta_0} w(t) \right] \, dt
\]

\[
 \leq \int_a^b |e^{-i\theta_0} w(t)| \, dt
\]

\[
 = \int_a^b |e^{-i\theta_0}||w(t)| \, dt
\]

\[
 = \int_a^b |w(t)| \, dt
\]

Since \( r_0 = \left| \int_a^b w(t) \, dt \right| \), we have

\[
\left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt.
\]

\( \square \)
Theorem 1. Let $C$ be a contour of length $L$, and suppose that $f(z)$ is piecewise continuous on $C$. If there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z$ on $C$ for which $f(z)$ is defined, then

$$\left| \int_C f(z) \, dz \right| \leq ML.$$

Proof. Let $z = z(t)$, $a \leq t \leq b$ be a parametric representation of $C$. Then

$$\left| \int_C f(z) \, dz \right| = \left| \int_a^b f[z(t)]z'(t) \, dt \right|$$

$$\leq \int_a^b |f[z(t)]z'(t)| \, dt$$

$$= \int_a^b |f([z(t)])||z'(t)| \, dt$$

$$\leq M \int_a^b |z'(t)| \, dt$$

$$= ML.$$

Examples.

(1) Let $C$ be the arc of the circle $|z| = 2$ that lies in the first quadrant. Show that

$$\left| \int_C \frac{dz}{z^2 + 1} \right| \leq \frac{\pi}{3}.$$

Let $f(z) = \frac{1}{z^2 + 1}$. First, find $M$ so that $|f(z)| \leq M$ on $C$. On $C$,

$$|f(z)| = \left| \frac{1}{z^2 + 1} \right|$$

$$= \frac{1}{|z^2 - (-1)|}$$

$$\leq \frac{1}{||z||^2 - 1}$$

(because $|z^2 - (-1)| \geq ||z||^2 - 1$)$$

= \frac{1}{2^2 - 1}$$

$$= \frac{1}{3}.$$

So,

$$\left| \int_C \frac{dz}{z^2 + 1} \right| \leq \frac{1}{3} \cdot \text{(length of $C$)} = \frac{1}{3} \cdot \frac{1}{4}(2\pi(2)) = \frac{\pi}{3} \cdot \checkmark$$

12
(2) Show that $\lim_{R \to \infty} \left| \int_{C_R} \frac{z + 4}{z^3 - 1} \, dz \right| = 0$, where $C_R$ is the path $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, for $R > 1$.

Let $f(z) = \frac{z + 4}{z^3 - 1}$. We need to find $M$ so that $|f(z)| \leq M$ on $C_R$.

$$\left| \frac{z + 4}{z^3 - 1} \right| = \frac{|z + 4|}{|z^3 - 1|} \leq \frac{|z| + 4}{||z^3| - 1|}$$

(because $|z^3 - 1| \geq ||z^3| - 1|$ and $|z + 4| \leq |z| + |4| = |z| + 4$)

$$= \frac{R + 4}{R^3 - 1}$$

$$= \frac{R + 4}{R^3 - 1}, \text{ where } R > 1.$$ 

Then,

$$\left| \int_{C_R} \frac{z + 4}{z^3 - 1} \, dz \right| \leq \frac{R + 4}{R^3 - 1} \cdot \text{(length of } C_R\text{)}$$

$$= \left( \frac{R + 4}{R^3 - 1} \right) \pi R$$

$$= \frac{\pi R^2 + 4\pi R}{R^3 - 1}.$$ 

Then,

$$0 \leq \lim_{R \to \infty} \left| \int_{C_R} \frac{z + 4}{z^3 - 1} \, dz \right| \leq \lim_{R \to \infty} \frac{\pi R^2 + 4\pi R}{R^3 - 1} = 0.$$ 

So,

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{z + 4}{z^3 - 1} \, dz \right| = 0. \checkmark$$

Exercise. Let $C$ denote the arc of the circle $|z| = 2$ that lies in the upper half plane. Show that

$$\left| \int_{C} \frac{z + 1}{z^2 - 2z + 3} \, dz \right| \leq 6\pi.$$ 

Solution. First, you must show that

$$|f(z)| = \left| \frac{z + 1}{z^2 - 2z + 3} \right| \leq \frac{|z| + 1}{||z - 1|^2 - 2|} \leq \frac{|z| + 1}{||z| - 1|^2 - 2|},$$

and then proceed as in the examples.
7 Antiderivatives – Section 44 of Brown and Churchill

Theorem 2. Suppose \( f(z) \) is a continuous function on a domain \( D \). If one of the following statements is true, then so are the others.

(a) \( f(z) \) has an antiderivative \( F(z) \) throughout \( D \).

(b) The integrals of \( f(z) \) along contours lying entirely in \( D \) and extending from any fixed point \( z_1 \) to any fixed point \( z_2 \) all have the same value, namely,

\[
\int_{z_1}^{z_2} f(z) \, dz = F(z)|_{z_1}^{z_2} = F(z_2) - F(z_1).
\]

(c) The integrals of \( f(z) \) around closed contours lying entirely in \( D \) have value 0.

Note: (b) says that the integral is path-independent.

Examples.

(1) Evaluate \( \int_C \frac{1}{z} \, dz \), where \( C \) is the circle of radius 1 centered at 2 traversed counterclockwise.

\[
\frac{1}{z} \text{ has antiderivative } \log z \text{ on the principal branch and } C \text{ lies entirely in the domain of definition, and since } C \text{ is a closed contour, } \int_C \frac{1}{z} \, dz = 0.
\]

(2) Evaluate \( \int_C z^2 \, dz \), where \( C \) is the semicircle of radius 1 centered at 2 traversed clockwise.

In this case, \( z^2 \) has an antiderivative, \( z^3 \), throughout the entire complex plane, so \( C \) is clearly in the domain of definition, and

\[
\int_C z^2 \, dz = \frac{z^3}{3}\bigg|_1^3 = \frac{26}{3}.
\]
8 Cauchy-Goursat Theorem – Section 46 of Brown and Churchill

**Theorem 3.** If a function \( f(z) \) is analytic at all points interior to, and on, a simple closed contour \( C \), then
\[
\int_C f(z) \, dz = 0.
\]

This theorem gives some indication of the power of analyticity.

**Note:** In order to apply the theorem, we must verify that the function to be integrated is analytic “inside” and on the contour \( C \).

We can now more easily re-do the first example from our discussion of antiderivatives.

**Example.** Evaluate \( \int_C \frac{1}{z} \, dz \), where \( C \) is the circle of radius 1 centered at 2 traversed counterclockwise.

\[
\int_C \frac{1}{z} \, dz = 0 \text{ because } f(z) = \frac{1}{z} \text{ is analytic on and inside } C. \text{ In fact, } \int_C z^n \, dz = 0 \text{ for } n \in \mathbb{Z}, \text{ since } f(z) = z^n \text{ is analytic on and inside } C.
\]

9 Simply Connected Domains – Section 48 of Brown and Churchill

**Definition.** A domain \( D \) is **simply connected** if every simple closed contour in \( D \) encloses only points in \( D \).

**Idea:** A simply connected region has no “holes.”

**Theorem 4.** If a function \( f \) is analytic throughout a simply connected domain \( D \), then
\[
\int_C f(z) \, dz = 0
\]
for every closed contour \( C \) lying in \( D \).

**Corollary 1.** A function \( f \) that is analytic throughout a simply connected domain \( D \) must have an antiderivative everywhere in \( D \).
10 Multiply Connected Domains – Section 49 of Brown and Churchill

A domain that is not simply connected is **multiply connected**.

**Theorem 5 (Cauchy-Goursat on Multiply Connected Domains).** Suppose that

(a) $C$ is a simple closed contour traversed counterclockwise; and

(b) $C_k$ ($k = 1, 2, \ldots, n$) are simple closed contours interior to $C$, all traversed clockwise, that are disjoint and whose interiors have no points in common.

If a function $f$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside $C$ and exterior to each $C_k$, then

$$\int_C f(z) \, dz + \sum_{k=1}^n \int_{C_k} f(z) \, dz = 0.$$ 

**Corollary 2 (Path Deformation Principle).** Let $C_1$ and $C_2$ denote positively oriented simple closed contours, where $C_1$ is interior to $C_2$. If a function $f$ is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.$$ 

**Idea:** If path $C_1$ can be continuously deformed into $C_2$, only passing through points at which $f$ is analytic, then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.$$
Proof. We apply the Cauchy-Goursat on Multiply Connected Domains theorem.

If we look at the above picture, we see that \( C_1 \) is counterclockwise, so application of the theorem gives

\[
\int_{C_2} f(z) \, dz + \int_{-C_1} f(z) \, dz = 0 \implies \int_{C_2} f(z) \, dz = -\int_{-C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]

\[\square\]

Examples.

(1) Let \( A \) be the region bounded by the \( x \)-axis and the upper half of the circle \(|z| = R\), where \( R > 0 \) is fixed. Let

\[
f(z) = \frac{e^{z^2}}{(2R - z)^2}.
\]

For each closed contour \( C \) in \( A \), show that \( \int_{C} f(z) \, dz = 0 \).

\( f(z) \) is analytic for all \( z \neq 2R \implies f(z) \) is analytic on the entire region \( A \), which is simply connected. Therefore, \( \int_{C} f(z) \, dz = 0 \).

(2) Evaluate \( \int_{C} \frac{1}{z} \, dz \), where \( C \) is the ellipse \( x^2 + 4y^2 = 1 \), traversed counterclockwise.

\[
\int_{C} \frac{1}{z} \, dz = \int_{C_0} \frac{1}{z} \, dz = 2\pi i \text{ (previously computed)}.
\]
(3) Evaluate \( \int_C \frac{3z - 2}{z^2 - z} \, dz \), where \( C \) is given by traversed counterclockwise.

We deform \( C \) into the “dumbell” given in your notes, where \( C_0 \) and \( C_1 \) are circles of radius \( r < 1 \).

\[
\int_C f(z) \, dz = \int_{C_0} f(z) \, dz + \int_{C_2} f(z) \, dz + \int_{C_1} f(z) \, dz + \int_{-C_2} f(z) \, dz \\
= \int_{C_0} f(z) \, dz + \int_{C_1} f(z) \, dz \\
= \int_{C_0} \frac{3z - 2}{z^2 - z} \, dz + \int_{C_1} \frac{3z - 2}{z^2 - z} \, dz \\
= \int_{C_0} \left( \frac{2}{z} + \frac{1}{z - 1} \right) \, dz + \int_{C_1} \left( \frac{2}{z} + \frac{1}{z - 1} \right) \, dz \\
= 2 \int_{C_0} \frac{1}{z} \, dz + \int_{C_0} \frac{1}{z - 1} \, dz + 2 \int_{C_1} \frac{1}{z} \, dz + \int_{C_1} \frac{1}{z - 1} \, dz
\]

Note that \( \frac{1}{z - 1} \) is analytic on and inside \( C_0 \), and \( \frac{1}{z} \) is analytic on and inside \( C_1 \), so

\[
\int_C f(z) \, dz = 2(2\pi i) + 0 + 2(0) + \int_{C_1} \frac{1}{z - 1} \, dz = 4\pi i + \int_0^{2\pi} \frac{1}{(re^{i\theta} + 1) - 1} \cdot ire^{i\theta} \, d\theta = 4\pi i + \int_0^{2\pi} i \, d\theta = 4\pi i + 2\pi i = 6\pi i.
\]
(4) Evaluate \( \int_C \frac{e^z}{z^2 - 9} \, dz \) on \( C : |z| = 2 \), traversed counterclockwise.

\[ f \text{ is analytic on and inside } C, \text{ since its only singularities are } z = -3 \text{ and } z = 3. \]

\[ \Rightarrow \int_C \frac{e^z}{z^2 - 9} \, dz = 0. \]

**Exercise.** Find \( \int_C \frac{1}{z - 2 - i} \, dz \), where \( C \) is the rectangle \( 0 \leq x \leq 4, 0 \leq y \leq 2 \), traversed counterclockwise.

**Solution.** Note that the only singularity is \( z = 2 + i \), which is inside the rectangle. So, let \( C_1 \) be the circle of radius \( r \) centered at \( z = 2 + i \), and use this to directly determine the value for the integral (i.e., as we did in Section 41). The correct answer is

\[ \int_C \frac{1}{z - 2 - i} \, dz = 2\pi i. \]

11 The Cauchy Integral Formula – Section 50 of Brown and Churchill

**Theorem 6.** Let \( f \) be analytic everywhere on and inside a simple closed contour \( C \), taken in the positive sense. If \( z_0 \) is any point interior to \( C \), then

\[ f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz. \quad (5) \]

Equation (5) is the **Cauchy integral formula**.

Rewriting equation (5) gives

\[ \int_C \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0), \quad (6) \]

which can be used to evaluate integrals.
Examples.

(1) Compute \( \int_C \frac{z}{(9 - z^2)(z + i)} \, dz \), where \( C \) is the circle \(|z| = 2\) traversed counterclockwise.

First, we note that \( \frac{z}{(9 - z^2)(z + i)} \) is analytic except where \((9 - z^2)(z+i) = 0\), or \( z = -i, 3, -3 \).

We see that the only singularity inside \( C \) is \( z_0 = -i \). Re-write the integral in the form

\[
\int_C \frac{f(z)}{z - z_0} \, dz,
\]
or

\[
\int_C \frac{z}{(9 - z^2)(z + i)} \, dz = \int_C \frac{(9 - z^2)}{z + i} \, dz.
\]

Then, \( f(z) = \frac{z}{9 - z^2} \), which is analytic on and inside \( C \), and \( z_0 = -i \) is inside \( C \).

\[
\int_C \frac{z}{(9 - z^2)z + i} \, dz = 2\pi i \left( \frac{z}{9 - z^2} \right) \bigg|_{z = -i} = 2\pi i \left( \frac{-i}{9 - (-i)^2} \right) = \frac{\pi}{5}.
\]

(2) Compute \( \int_C \frac{e^z + \sin z}{z} \, dz \), where \( C \) is the circle \(|z - 2| = 3\) traversed counterclockwise.

Here, the function to be integrated is analytic except at \( z = 0 \), so \( f(z) = e^z + \sin z \) and \( z_0 = 0 \).

\( f(z) \) is analytic on and inside \( C \) and \( z_0 = 0 \) is inside \( C \), so

\[
\int_C \frac{e^z + \sin z}{z} \, dz = 2\pi i \left( e^z + \sin z \right) \bigg|_{z = 0} = 2\pi i.
\]
(3) Evaluate \( \int_C \frac{\cos z}{z^2 - 4} \, dz \), where \( C \) is given by

traversed counterclockwise.

\[
\frac{\cos z}{z^2 - 4} = \frac{\cos z}{(z + 2)(z - 2)},
\]

which is analytic for \( z \neq \pm 2 \). \( z_0 = 2 \) is the only singularity inside \( C \), so

\[ f(z) = \frac{\cos z}{z + 2}. \]

\( f(z) \) is analytic on and inside \( C \), and \( z_0 = 2 \) is inside \( C \), so

\[
\int_C \frac{\cos z}{z^2 - 4} \, dz = \int_C \frac{\cos z}{z - 2} \, dz
\]

\[
= 2\pi i \left( \frac{\cos z}{z + 2} \right) \bigg|_{z=2}
\]

\[
= \frac{\pi i}{2} \cos 2.
\]

(4) Evaluate \( \int_C \frac{z^2 e^z}{2z - 1} \, dz \), where \( C \) is the circle \( |z| = 1 \) traversed clockwise.

![Diagram of circle C with points labeled -1 and 1]
First, we need to rewrite the integrand in the form \( \frac{f(z)}{z - z_0} \). Note that

\[
\frac{z^2 e^z}{2z - 1} = \frac{z^2 e^z}{z - \frac{1}{2}},
\]

which is analytic for \( z \neq \frac{1}{2} \), so define

\[
f(z) = \frac{z^2 e^z}{2} \quad \text{and} \quad z_0 = \frac{1}{2}.
\]

Then, \( f(z) \) is analytic on and inside \( C \), and \( z_0 \) is inside \( C \). Therefore,

\[
\int_C \frac{z^2 e^z}{2z - 1} \, dz = -\int_{-C} \frac{z^2 e^z}{z - \frac{1}{2}} \, dz
\]

(This is done because \( C \) is traversed clockwise, not counterclockwise.)

\[
= -2\pi i \left( \frac{z^2 e^z}{2} \right) \bigg|_{z=\frac{1}{2}}
\]

\[
= -\frac{\pi ie^{\frac{1}{2}}}{4}.
\]

**Exercise.** Evaluate \( \int_C \frac{\cosh z}{z^2 - 4z + 3} \, dz \), where \( C \) is the circle \(|z| = 2\) traversed counterclockwise.

**Solution.** The given integrand is analytic except where \( z^2 - 4z + 3 = 0 \), or except at \( z = 1, 3 \). Since only \( z = 1 \) is inside \( C \), we may set \( f(z) = \frac{\cosh z}{z - 1} \). Then, verify that \( f(z) \) and \( z_0 \) satisfy the requirements for the Cauchy integral formula, and apply it to obtain

\[
\int_C \frac{\cosh z}{z^2 - 4z + 3} \, dz = -\pi \cosh 1i.
\]

### 12 An Extension of the Cauchy Integral Formula

**Section 51 of Brown and Churchill**

**Theorem 7 (Cauchy Integral Formula for Derivatives).** Let \( f \) be analytic on a region \( A \). Then all derivatives of \( f \) exist on \( A \). Furthermore, for any simple closed contour \( C \) in \( A \), taken in the positive sense, if \( z_0 \) is any point interior to \( C \), then

\[
\frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz = f^{(n)}(z_0).
\]
Rewriting equation (7), we obtain

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

which can be used to evaluate integrals.

**Examples.**

(1) Compute $\int_C e^{5z} \frac{z^3}{z^2} \, dz$, where $C$ is the circle $|z| = 1$ traversed counterclockwise.

The integrand is already in the form for which we may apply equation (8) with $f(z) = e^{5z}$, $z_0 = 0$, and $n = 2$. Since $f(z)$ is entire, $f(z)$ is analytic on and inside $C$, and $z_0 = 0$ is inside $C$, so

$$\int_C e^{5z} \frac{z^3}{z^2} \, dz = \frac{2\pi i}{2!} f''(0)$$

$$= \pi i \cdot 25e^{5(0)}$$

$$= \frac{25\pi i}{2}.$$

(2) Compute $\int_C \frac{2z + 1}{z(z - 1)^2} \, dz$, where $C$ is given by

$$\int_C = \int_{C_1} + \int_{C_2} = \int_{C_1} - \int_{-C_2}$$

$$\int_{C_1} : f(z) = \frac{2z + 1}{z}, \quad z_0 = 1, \quad n = 1. \quad f \text{ is analytic on and inside } C_1,$$

$$\int_{C_1} \frac{2z + 1}{z(z - 1)^2} \, dz = \frac{2\pi i}{1!} \cdot \left. \frac{d}{dz} \left( \frac{2z + 1}{z} \right) \right|_{z=1}$$

$$= 2\pi i (-1)$$

$$= -2\pi i.$$
\[ \int_{-C_2} : f(z) = \frac{2z + 1}{(z - 1)^2}, \quad z_0 = 0. \]  
\[ f \text{ is analytic on and inside } -C_2, \text{ so} \]
\[ \int_{-C_2} \frac{2z + 1}{z} \, dz = 2\pi i \cdot \frac{2z + 1}{(z - 1)^2} \bigg|_{z=0} \]
\[ = 2\pi i (1) \]
\[ = 2\pi i. \]

Therefore,
\[ \int_{C} \frac{2z + 1}{z(z - 1)^2} \, dz = -2\pi i - 2\pi i = -4\pi i. \]

(3) Evaluate \( \int_{C} (z - z_0)^k \, dz \), where \( C \) is the circle \(|z - z_0| = r\) traversed counterclockwise and \( k \in \mathbb{Z} \).

We must consider three cases: \( k \geq 0 \), \( k < -1 \), and \( k = -1 \).

\( k \geq 0 \): \( \int_{C} (z - z_0)^k \, dz = 0 \) by the Cauchy-Goursat theorem, since \( C \) is a simple closed contour and \( f(z) = (z - z_0)^k \) is analytic on and inside \( C \).

\( k < -1 \): Let \( k = -m \), where \( m > 1 \). Then, we look at
\[ \int_{C} \frac{dz}{(z - z_0)^m} \, dz \implies f(z) = 1, n = m - 1. \]
Clearly, \( f \) is analytic on and inside \( C \) and \( z_0 \) is inside \( C \), so
\[ \int_{C} \frac{dz}{(z - z_0)^m} \, dz = \frac{2\pi i}{(m - 1)!} \cdot \frac{d^{m-1}}{dz^{m-1}}(1) \bigg|_{z=z_0} = 0. \]

\( k = -1 \): Again, we obtain \( f = 1 \) and \( f \) is clearly analytic on and inside \( C \) and \( z_0 \) is inside \( C \), so
\[ \int_{C} \frac{dz}{z - z_0} \, dz = 2\pi i \cdot f(z_0) = 2\pi i. \]

13 Some Consequences of Cauchy’s Integral Formula – Sections 52-54 of Brown and Churchill

1) Cauchy’s Inequalities

**Theorem 8.** Let \( f \) be analytic on a region \( A \) and let \( C \) be defined by \(|z - z_0| = R, \ z_0 \in A\). Assume that the disk given by \(|z - z_0| < R\) also lies in \( A \). Suppose \(|f(z)| \leq M\) for all \( z \) on \( C \). Then, for any \( n = 0, 1, 2, \ldots \),
\[ |f^{(n)}(z_0)| \leq \frac{n!}{R^n} \cdot M. \]
Proof. By Cauchy’s integral formula,

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz
\]

\[
\Rightarrow |f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right|
\]

\[
\leq \frac{n!}{2\pi} \int_C \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| |dz|
\]

\[
\leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot \text{(length of } C\text{)}
\]

\[
= \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R
\]

\[
= \frac{n!}{R^n} \cdot M.
\]

\[\square\]

2) Liouville’s Theorem

**Theorem 9 (Liouville’s Theorem).** If a function \( f \) is entire and bounded in the complex plane, then \( f(z) \) is constant throughout the plane.

**Proof.** By the Cauchy Inequalities, if \( n = 1 \), then

\[
|f'(z_0)| \leq \frac{M}{R}.
\]

Fix \( z_0 \) and take the limit as \( R \to \infty \) of both sides

\[
\Rightarrow |f(z_0)| = 0.
\]

This is true for every \( z_0 \in \mathbb{C} \), so \( f \) is constant. \[\square\]

3) Fundamental Theorem of Algebra

**Theorem 10 (Fundamental Theorem of Algebra).** Any polynomial \( p(z) = a_0 + a_1 z + \cdots + a_n z^n \) \((a_n \neq 0)\) of degree \( n \geq 1 \) has at least one zero; i.e., there is at least one point \( z_0 \) such that \( p(z_0) = 0 \).

4) Morera’s Theorem

**Theorem 11 (Morera’s Theorem).** Let \( f \) be continuous on a region \( A \) and suppose \( \int_C f(z) \, dz = 0 \) for every closed contour \( C \) lying in \( A \). Then \( f \) is analytic on \( A \).

5) The Maximum Modulus Principal – Section 54 of Brown and Churchill

**Lemma 2.** Suppose that \( |f(z)| \leq |f(z_0)| \) at each point \( z \) in some neighborhood \( |z - z_0| < \epsilon \) in which \( f \) is analytic. Then \( f(z) \) has the constant value \( f(z_0) \) throughout that neighborhood.
**Theorem 12 (Maximum Modulus Principle).** Let \( R \) be a closed, connected, bounded region. If \( f \) is continuous on \( R \) and analytic and not constant in the interior of \( R \), then \( |f(z)| \) achieves its maximum value on the boundary of \( R \), and never in the interior of \( R \).

The theorem is proved using repeated applications of the lemma.

**Note:** It can be proved that if a function satisfies the hypotheses of the Maximum Modulus Principle and \( f(z) \neq 0 \) for all \( z \in R \), then \( |f(z)| \) achieves its minimum value on the boundary of \( R \) and never in the interior of \( R \).

**Example.** Find the maximum of \( |\cos z| \) on \([0, 2\pi] \times [0, 2\pi]\)

- Since \( \cos z \) is entire, the Maximum Modulus Principle may be applied.
- So, we know that the maximum of \( |\cos z| \) occurs on the boundary of the square.
- Therefore, we must determine the maximum value of \( |\cos z| \) along each of the four boundary segments.

\[
\cos(z) = \cos(x + iy) \\
= \cos x \cos(iy) - \sin x \sin(iy) \\
= \cos x \cosh y - i \sin x \sinh y,
\]

since

\[
\cos(iy) = \frac{e^{-y} + e^y}{2} = \cosh y \\
\sin(iy) = \frac{e^{-y} - e^y}{2i} = i \sinh y.
\]

Then,

\[
|\cos z|^2 = (\cos x \cosh y)^2 + (-\sin x \sinh y)^2 \\
= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
= \cos^2 x \cosh^2 y + \sin^2 x \cosh^2 y - \sin^2 x \cosh y + \sin^2 x \sinh^2 y \\
= (\cos^2 x + \sin^2 x) \cosh^2 y - \sin^2 x (\cosh^2 y - \sinh^2 y) \\
= \cosh^2 y - \sin^2 x.
\]

- On the boundary \( x = 0 \), \( |\cos z|^2 \) has maximum \( \cosh^2(2\pi) \).
- On the boundary \( y = 0 \), \( |\cos z|^2 \) has maximum 1.
- On the boundary \( x = 2\pi \), \( |\cos z|^2 \) has maximum \( \cosh^2(2\pi) \).
- On the boundary \( y = 2\pi \), \( |\cos z|^2 \) has maximum \( \cosh^2(2\pi) \).

So, the maximum of \( |\cos z| \) on \([0, 2\pi] \times [0, 2\pi]\) is \( \cosh(2\pi) \).