p. 11: 26

Suppose the system below is consistent for all possible values of $f$ and $g$. What can you say about the coefficients $c$ and $d$? Justify your answer.

\[
\begin{align*}
2x_1 + 4x_2 &= f \\
cx_1 + dx_2 &= g
\end{align*}
\]

**Solution:** Write the augmented matrix and put it in echelon form.

\[
\begin{bmatrix}
2 & 4 & f \\
c & d & g
\end{bmatrix}
\xrightarrow{r_2 \rightarrow r_2 - \frac{c}{2} r_1}
\begin{bmatrix}
2 & 4 & f \\
0 & d - 2c & g - \frac{fc}{2}
\end{bmatrix}.
\]

For the system to be consistent for all values of $f$ and $g$, we need $d - 2c \neq 0$. (Otherwise, we obtain the equation $0 = g - \frac{fc}{2} \neq 0$ for all values of $f$ and $g$.)

p. 22: 10

Find the general solution of the system whose augmented matrix is

\[
\begin{bmatrix}
1 & -2 & -1 & 4 \\
-2 & 4 & -5 & 6
\end{bmatrix}.
\]

**Solution:** First, put the matrix in echelon form.

\[
\begin{bmatrix}
1 & -2 & -1 & 4 \\
-2 & 4 & -5 & 6
\end{bmatrix}
\xrightarrow{r_2 \rightarrow r_2 + 2r_1}
\begin{bmatrix}
1 & -2 & -1 & 4 \\
0 & 0 & -7 & 6
\end{bmatrix}.
\]

Then the first and third columns are the pivot columns, so $x_1$ and $x_3$ are basic variables, and $x_2$ is a free variable. The resulting system is

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 4 \\
-7x_3 &= 14 \implies x_3 = -2.
\end{align*}
\]
Then
\[ x_1 = 4 + 2x_2 + x_3 \]
\[ = 4 + 2x_2 - 2 \]
\[ = 2 + 2x_2. \]

Therefore, the solution is
\[
\begin{align*}
  x_1 &= 2 + 2x_2 \\
  x_2 &\text{ is free} \\
  x_3 &= -2
\end{align*}
\]

p. 33: 27

A mining company has two mines. One day’s operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kilograms of silver, while one day’s operation at mine #2 produces ore that contains 40 metric tons of copper and 380 kilograms of silver. Let \( \mathbf{v}_1 = \begin{bmatrix} 30 \\ 600 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 40 \\ 380 \end{bmatrix} \). Then \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) represent the “output per day” of mine #1 and mine #2, respectively.

(a) What physical interpretation can be given to the vector \( 5\mathbf{v}_1 \)?

**Solution:** Since \( \mathbf{v}_1 \) represents the output per day of mine #1, \( 5\mathbf{v}_1 \) represents the output of mine #1 after five days.

(b) Suppose the company operates mine #1 for \( x_1 \) days and mine #2 for \( x_2 \) days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kilograms of silver. Do not solve the equation.

**Solution:** Let \( \mathbf{b} = \begin{bmatrix} 240 \\ 2824 \end{bmatrix} \) be the vector representing the total desired output. then, we want to find \( x_1 \) and \( x_2 \) so that

\[ x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}, \text{ or } x_1\begin{bmatrix} 30 \\ 600 \end{bmatrix} + x_2\begin{bmatrix} 40 \\ 380 \end{bmatrix} = \begin{bmatrix} 240 \\ 2824 \end{bmatrix}. \]

(c) Solve the equation in (b).

**Solution:** The vector equation has the same solution as the system represented by the augmented matrix
\[
\begin{bmatrix}
  30 & 40 & | & 240 \\
  600 & 380 & | & 2824
\end{bmatrix}.
\]
Solve:
\[
\begin{bmatrix}
30 & 40 & | & 240 \\
600 & 380 & | & 2824
\end{bmatrix}
\xrightarrow{r_2 \rightarrow r_2 - 20r_1}
\begin{bmatrix}
30 & 40 & | & 240 \\
0 & -420 & | & -1976
\end{bmatrix}.
\]

The resulting system is
\[
\begin{align*}
30x_1 + 40x_2 &= 240 \\
-420x_2 &= -1976 \implies x_2 = \frac{494}{105} \text{ days} \approx 5 \text{ days.}
\end{align*}
\]

\[
x_1 = 8 - \frac{4}{3}x_2
\]

\[
= 8 - \frac{4}{3} \left( \frac{494}{105} \right)
\]

\[
= \frac{533}{315} \text{ days} \approx 2 \text{ days.}
\]

So, mine #1 should be operated for 5 days and mine #2 should be operated for 2 days.

**p. 42: 34**

Let $A$ be a $3 \times 4$ matrix, let $v_1$ and $v_2$ be vectors in $\mathbb{R}^3$, and let $w = v_1 + v_2$. Suppose $v_1 = Au_1$ and $v_2 = Au_2$ for some vectors $u_1$ and $u_2$ in $\mathbb{R}^4$. What fact allows you to conclude that $Ax = w$ is consistent?

**Solution:** Since $v_1 = Au_1$ and $v_2 = Au_2$, $Au_1 + Au_2 = v_1 + v_2$. So,

\[
A(u_1 + u_2) = Au_1 + Au_2
\]

\[
= v_1 + v_2
\]

\[
= w.
\]

Therefore, $Ax = w$ has at least one solution (i.e., $u_1 + u_2$), and so is consistent.