Homework # 6 Solutions

Math 152, Fall 2014
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p. 196-7: 6, 8, 20, 22 (Section 4.1)

For problems 6 and 8, determine if the given set is a subspace of $P_n$ for an appropriate value of $n$. Justify your answers.

6. All polynomials of the form $p(t) = a + t^2$, where $a \in \mathbb{R}$.

**Solution:**

(a) The zero polynomial is given by $0 = 0 + 0t + 0t^2 = 0 + 0t^2$. The coefficient of $t^2$ is 0, not 1, so 0 is not in the set.

Therefore, this set is not a subspace of $P_2$.

8. All polynomials in $P_n$ such that $p(0) = 0$.

**Solution:** We will call this set $H$ for ease of reference.

(a) The zero polynomial in $P_n$ is $0 = 0 + 0t + \cdots + 0t^n$. Evaluating this polynomial at $t = 0$ gives us 0. Therefore, 0 $\in H$.

(b) Let $p \in H$. Then $p(0) = 0$.

Let $q \in H$. Then $q(0) = 0$.

We then have that

$$(p + q)(0) = p(0) + q(0)$$

$$= 0 + 0 = 0.$$

Therefore, $p + q \in H$.

(c) Let $p \in H$ and let $c$ be a scalar. Then

$$(cp)(0) = c \cdot p(0)$$

$$= c \cdot 0 = 0.$$

Therefore, $cp \in H$.

Since $H$ satisfies properties (a), (b), and (c), $H$ is a subspace of $P_n$. 

20. The set of all continuous real-valued functions defined on a closed interval \([a, b]\) in \(\mathbb{R}\) is denoted by \(C[a, b]\). This set is a subspace of the vector space of all real-valued functions defined on \([a, b]\).

(a) What facts about continuous functions should be proved in order to demonstrate that \(C[a, b]\) is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

**Solution:** We need the following facts:

(i) The constant function \(f(t) = 0\) is continuous.

(ii) Sums of continuous functions are continuous.

(iii) A constant multiple of continuous functions is continuous.

(b) Show that \(\{f \in C[a, b] : f(a) = f(b)\}\) is a subspace of \(C[a, b]\).

**Solution:** Let \(H = \{f \in C[a, b] : f(a) = f(b)\}\).

(a) The zero function \(f(t) = 0\) for all \(t\) is in \(H\) since \(f(a) = 0 = f(b)\).

(b) Let \(g \in H\). Then \(g(a) = g(b)\).

Let \(h \in H\). Then \(h(a) = h(b)\).

So,

\[
(g + h)(a) = g(a) + h(a) = g(b) + h(b) = (g + h)(b).
\]

Therefore, \(g + h \in H\).

(c) Let \(g \in H\) and let \(c\) be a scalar. Then,

\[
(cg)(a) = c \cdot g(a) = c \cdot g(b) = (cg)(b).
\]

Therefore, \(cg \in H\).

Therefore, \(H\) is a subspace of \(C[a, b]\).

22. Let \(F\) be a fixed \(3 \times 2\) matrix and let \(H\) be the set of matrices \(A \in M_{2 \times 4}\) with the property that \(FA = 0\) (the zero matrix in \(M_{3 \times 4}\). Determine if \(H\) is a subspace of \(M_{2 \times 4}\).

**Solution:**

(a) \(0 \in H\) since \(F0 = 0\).

(b) Let \(A \in H \implies FA = 0\).

Let \(B \in H \implies FB = 0\).

Then,

\[
F(A + B) = FA + FB = 0 + 0 = 0.
\]

Therefore, \(A + B \in H\).
(c) Let \( A \in H \) and let \( c \) be a scalar. Then,

\[
F(cA) = c(FA) = c(0) = 0.
\]

Therefore, \( cA \in H \).

Therefore, \( H \) is a subspace of \( M_{2\times4} \).

p. 206-7: 28, 33(a), (c), (d) (Section 4.2)

28. Consider the following two systems of equations:

\[
\begin{align*}
5x_1 + x_2 - 3x_3 &= 0 \\
-9x_1 + 2x_2 + 5x_3 &= 1 \\
4x_1 + x_2 - 6x_3 &= 9
\end{align*}
\]

\[
\begin{align*}
5x_1 + x_2 - 3x_3 &= 0 \\
-9x_1 + 2x_2 + 5x_3 &= 5 \\
4x_1 + x_2 - 6x_3 &= 45
\end{align*}
\]

It can be shown that the first system has a solution. Use this fact and the theory from this section to explain why the second system must also have a solution.

**Solution:** Consider the matrix equation of the first system:

\[
\begin{bmatrix}
5 & 1 & -3 \\
-9 & 2 & 5 \\
4 & 1 & -6
\end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}.
\]

The matrix equation for the second system is

\[
\begin{bmatrix}
5 & 1 & -3 \\
-9 & 2 & 5 \\
4 & 1 & -6
\end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}.
\]

Let

\[
A = \begin{bmatrix}
5 & 1 & -3 \\
-9 & 2 & 5 \\
4 & 1 & -6
\end{bmatrix}.
\]

The first equation tells us that \( \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} \in \text{Col} \ A \). Since \( \text{Col} \ A \) is a subspace of \( \mathbb{R}^3 \) and since

\[
\begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix},
\]

we know that \( \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} \in \text{Col} \ A \).
33. Let $M_{2 \times 2}$ be the vector space of all $2 \times 2$ matrices, and define $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ by $T(A) = A + A^T$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Show that $T$ is a linear transformation.

**Solution:** Let $A, B \in M_{2 \times 2}$ and let $c$ be a scalar.

(i) $T(A + B) = T(A) + T(B)$

$$T(A + B) = (A + B) + (A + B)^T$$
$$= (A + B) + A^T + B^T$$
$$= (A + A^T) + (B + B^T)$$
$$= T(A) + T(B).$$

(ii) $T(cA) = cT(A)$

$$T(cA) = (cA) + (cA)^T$$
$$= cA + cA^T$$
$$= c(A + A^T)$$
$$= cT(A).$$

Therefore, $T$ is a linear transformation.

(c) Show that the range of $T$ is the set of $B$ in $M_{2 \times 2}$ such that $B^T = B$.

**Solution:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then, the range of $T$ is the set of all matrices $B$ such that $B = T(A)$. We see that

$$B = T(A) = A + A^T = \begin{bmatrix} 2a \\ c + b \\ b + c \\ 2d \end{bmatrix} = B^T.$$ 
Therefore, the range of $T$ is the set of matrices in $M_{2 \times 2}$ such that $B^T = B$.

(d) Describe the kernel of $T$.

**Solution:** We seek $A \in M_{2 \times 2}$ so that $T(A) = 0$, or $A + A^T = 0$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

and

$$A + A^T = \begin{bmatrix} 2a & b + c \\ c + b & 2d \end{bmatrix}.$$ 
We need to solve for $a, b, c, d$.

$$\begin{bmatrix} 2a & b + c \\ c + b & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
Since two matrices are equal if their corresponding entries are equal, we have

\[
\begin{align*}
2a &= 0 \implies a = 0 \\
b + c &= 0 \implies b = -c \\
c + b &= 0 \implies c = -b \\
2d &= 0 \implies d = 0.
\end{align*}
\]

Therefore, \( A \in \ker A \) if

\[
A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

So,

\[
\ker A = \text{Span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.
\]