Math 111, Fall 2014 - Homework # 10

Due Thursday, November 20, 2014, by 4:30 p.m.

Prove each of the following with either induction, strong induction, or proof by smallest counterexample.

1. For every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Solution:

2. If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

   Solution:

3. For any integer $n \geq 0$, it follows that $9 \mid (4^{3n} + 8)$.

   Solution:

4. Suppose that $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Then

   $$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

   Solution:

5. For every natural number n, it follows that $2^n + 1 \leq 3^n$.

   Solution:

6. Prove that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$ for every $n \in \mathbb{N}$.

   Solution:

7. Prove that $\displaystyle\sum_{k=s}^{N} \binom{k}{s} = \binom{N+1}{s+1}$ for all natural numbers s and N such that $N \geq s$.

   Hint: Prove this by induction on N. You may find an equality from Section 3.4 useful, as well.

   Solution:

8. Let F_n be the n^{th} term of the Fibonacci sequence. Then $\displaystyle\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

   Solution: