Extra Credit Solutions

Math 111, Fall 2014
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1. The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(m, n) = (5m + 4n, 4m + 3n)$ is bijective. Find its inverse.
   You do <u>not</u> need to prove that the function is bijective.

   Solution: Write $f(m, n) = (a, b)$. Interchange the variables to obtain

$$(m, n) = f(a, b) = (5a + 4b, 4a + 3b).$$

Then, we solve for a and b in terms of m and n. So we have

$$\begin{aligned} 5a + 4b &= m \\ 4a + 3b &= n. \end{aligned}$$

This is a system of two equations in two unknowns, which we may solve without too much trouble. Multiply the first equation by 4 and the second equation by 5 to obtain

$$\begin{aligned} 20a + 16b &= 4m \\ 20a + 15b &= 5n. \end{aligned}$$

Subtracting the new first equation from the second gives the system

$$\begin{aligned} 20a + 16b &= 4m \\ -b &= 5n - 4m \implies b = 4m - 5n. \end{aligned}$$

Finally, substitute the expression for b into the first equation to obtain

$$\begin{aligned} 20a + 16(4m - 5n) &= 4m \\ 20a + 64m - 80n &= 4m \\ 20a &= -60m + 80n \\ a &= -3m + 4n. \end{aligned}$$

Therefore,

$$f^{-1}(m, n) = (a, b) = (-3m + 4n, 4m - 5n).$$

2. Let $A = \{x \in \mathbb{R} : x \geq 1\}$ and $B = \{x \in \mathbb{R} : x > 0\}$. For each function below, determine $f(A)$, $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(\{1\})$.

   (a) $f : \mathbb{R} \to B$ defined by $f(x) = e^{x^3+1}$

   (b) $f : \mathbb{R} \to \mathbb{R}$ defind by $f(x) = x^2$

   Solution:

   (a) Note that $f(x)$ is an increasing function, since e^u is increasing when $u > 0$, and that $f(1) = e^{1^3+1} = e^2$. So, $f(A) = \{x \in \mathbb{R} : x \geq e^2\}$.

   To determine the remaining answers, note that we can prove that f is bijective (exercise for you), so f is invertible. We will determine f^{-1} before completing this problem. Since $(f \circ f^{-1})(x) = x$ for $x \in B$, we have that

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = e^{(f^{-1}(x))^3+1} = x.$$

   Let $y = f^{-1}(x)$ and solve for y. We have

$$\begin{aligned} x &= e^{y^3+1} \\ \ln x &= y^3 + 1 \quad \text{(take the natural log of both sides, which we can do since } x \in B.) \\ y^3 &= \ln x - 1 \\ y &= (\ln x - 1)^{\frac{1}{3}}. \end{aligned}$$

   Therefore,

$$f^{-1}(x) = (\ln x - 1)^{\frac{1}{3}}.$$

   Since $A \subseteq B$, we can determine $f^{-1}(A)$. On B, the function $\ln x$ is increasing and so $f^{-1}(x)$ is increasing. Since $f^{-1}(1) = (\ln 1 - 1)^{\frac{1}{3}}$, we have that $f^{-1}(1) = -1$ and so $f^{-1}(A) = \{x \in \mathbb{R} : x \geq -1\}$. If $0 < x < 1$, then $\ln x$ is negative and $\lim_{x \to 0} \ln x = -\infty$. Therefore, $f^{-1}(B) = \mathbb{R}$. Finally, $f^{-1}(\{1\}) = \{-1\}$. To summarize,

$$\begin{aligned} f(A) &= \{x \in \mathbb{R} : x \geq e^2\}, \\ f^{-1}(A) &= \{x \in \mathbb{R} : x \geq -1\}, \\ f^{-1}(B) &= \mathbb{R}, \\ f^{-1}(\{1\}) &= \{-1\}. \end{aligned}$$

   (b) First, consider how the function $f(x)$ behaves on A. If $x \geq 1$, we have that $f(x) = x^2$ is increasing (since $f'(x) = 2x$). Therefore, $f(x)$ takes its minimum value on A at $x = 1$ and $f(1) = 1^2 = 1$. Therefore, $f(A) = \{x \in \mathbb{R} : x \geq 1\} = [1, \infty) = A$.

   To find the required inverse images, we first note that $A = [1, \infty)$ and $B = (0, \infty)$. Now, consider $A = [1, \infty)$. If $f(x) = 1$, then $x = -1$ or $x = 1$. For all $f(x) \geq 1$, we have that $x = \pm\sqrt{f(x)}$. Since $f(x)$ is increasing when $x > 0$ and decreasing for $x < 0$, we have that

$$f^{-1}(A) = \{x \in \mathbb{R} : x \leq -1\} \cup \{x \in \mathbb{R} : x \geq 1\} = (-\infty, -1] \cup [1, \infty).$$

Next, consider $B = (0, \infty)$. Since when $f(x) = 0$, we have that $x = 0$ and since $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, we have that $f^{-1}(B) = \{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > 0\} = (-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\}$. Finally, if $f(x) = 1$, then $x = \pm 1$. Therefore, $f^{-1}(\{1\}) = \{-1, 1\}$. To summarize,

$$\begin{aligned} f(A) &= [1, \infty) = A, \\ f^{-1}(A) &= (-\infty, -1] \cup [1, \infty), \\ f^{-1}(B) &= \mathbb{R} - \{0\}, \\ f^{-1}(\{1\}) &= \{-1, 1\}. \end{aligned}$$

3. Given a function $f : C \to Z$ and sets $A, B \subseteq C$ and $X, Y \subseteq Z$.
   (a) Prove or dispove: $f(A \cap B) = f(A) \cap f(B)$.
   (b) Prove or disprove: $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.

Solution:

(a) This statement is false, because $f(A) \cap f(B) \not\subseteq f(A \cap B)$.
Counterexample. Let $C = \mathbb{Z}$, $A = \{x \in \mathbb{Z} : x \geq 0\}$, and $B = \{x \in \mathbb{Z} : x \leq 0\}$, and let $Z = \mathbb{Z}$. Define $f : C \to Z$ by $f(x) = x^2$. Then $f(A) = A$ since $f(x)$ is increasing for $x \geq 0$, with its minimum at $x = 0$ and $f(0) = 0^2 = 0$. On B, while $x \leq 0$, $f(x)$ is increasing, taking its minimum value at $x = 0$. Since $f(0) = 0$, $f(B) = \{x \in \mathbb{Z} : x \geq 0\} = A$. Therefore, $f(A) \cap f(B) = A$. But, since $A \cap B = \{0\}$, $f(A \cap B) = f(\{0\}) = \{0\} \neq A$.

(b) This is a true statement.

Proof. We first show that $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$. Suppose $x \in f^{-1}(X \cap Y)$. This means that $f(x) \in X \cap Y$. Therefore, $f(x) \in X$ and $f(x) \in Y$. If $f(x) \in X$, then $x \in f^{-1}(X)$ and if $f(x) \in Y$, then $x \in f^{-1}(Y)$. Therefore, $x \in f^{-1}(X)$ and $x \in f^{-1}(Y)$, so $x \in f^{-1}(X) \cap f^{-1}(Y)$ and $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$.
Next, we show that $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$. Suppose $y \in f^{-1}(X) \cap f^{-1}(Y)$. Then $y \in f^{-1}(X)$ and $y \in f^{-1}(Y)$. This means that $f(y) \in X$ and $f(y) \in Y$. Therefore, $f(y) \in X \cap Y$. This means that $y \in f^{-1}(X \cap Y)$. Therefore, $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$.
Since $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$ and $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$, $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$. □