Homework # 3 Solutions

Math 111, Fall 2014
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1. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1, r+1]$, and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

   (a) $\displaystyle\bigcup_{r \in S} A_r$ and $\displaystyle\bigcap_{r \in S} A_r$

   (b) $\displaystyle\bigcup_{r \in S} B_r$ and $\displaystyle\bigcap_{r \in S} B_r$

   (c) $\displaystyle\bigcup_{r \in S} C_r$ and $\displaystyle\bigcap_{r \in S} C_r$.

   Solution:

   (a) $A_1 = \{1, \},\ A_2 = \{2\},$ and $A_4 = \{16\}$, so $\displaystyle\bigcup_{r \in S} A_r = \{1, 4, 16\}$ and $\displaystyle\bigcap_{r \in S} A_r = \varnothing$

   (b) $B_1 = [0, 1],\ B_2 = [1, 3],$ and $B_4 = [3, 5]$, so $\displaystyle\bigcup_{r \in S} B_r = [0, 5]$ and $\displaystyle\bigcap_{r \in S} B_r = \varnothing$

   (c) $C_1 = (1, \infty),\ C_2 = (2, \infty),$ and $C_4 = (4, \infty)$.
   Therefore, $\displaystyle\bigcup_{r \in S} C_r = (1, \infty)$ and $\displaystyle\bigcap_{r \in S} C_r = (4, \infty)$.

2. For $r \in \mathbb{R}^+$ ($\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$), let $A_r = \{x \in \mathbb{R} : |x| < r\}$. Determine $\displaystyle\bigcup_{r \in \mathbb{R}^+} A_r$ and $\displaystyle\bigcap_{r \in \mathbb{R}^+} A_r$.

   Solution: $A_r = \{x \in \mathbb{R} : |x| < r\} = \{x \in \mathbb{R} : -r < x < r\} = (-r, r)$. So, as r increases, the intervals get larger and larger. Therefore, $\displaystyle\bigcup_{r \in \mathbb{R}^+} A_r = (-\infty, \infty) = \mathbb{R}$. As r gets smaller and smaller, the intervals are decreasing in size. Since $r > 0$, $0 \in (-r, r)$ for all r. Therfore, $\displaystyle\bigcap_{r \in \mathbb{R}^+} A_r = \{0\}$

3. Determine which of the following are statements. For statements, determine if they are true or false.
   (i) Every even integer is a real number.
   (ii) $\mathbb{N} \notin P(\mathbb{N})$.
   (iii) The integer x is divisible by 5.
   (iv) $\varnothing = \{\varnothing\}$.

   Solution:

   (i) Every even integer is a real number. – Statement. True, since $\mathbb{Z} \subseteq \mathbb{R}$.
   (ii) $\mathbb{N} \notin P(\mathbb{N})$. – Statement. False, becuase the power set contains the original set as an element.
   (iii) The integer x is divisible by 5. – Not a statement.
   (iv) $\varnothing = \{\varnothing\}$. – Statement. False (see Chapter 1).

4. Express each statement or open sentence in one of the forms $P \wedge Q$, $P \vee Q$, or $\sim P$. Make sure to state exactly what statements P and Q stand for.
   (i) The matrix A is not invertible.
   (ii) $x < y$
   (iii) At least one of the numbers x and y equals 0.
   (iv) $x \in A \cap B$

   Solution:

   (i) P: The matrix A is invertible. The given statement, then, is $\sim P$.
   (ii) P: $x \geq y$. The given statement, then, is $\sim P$.
   (iii) P: x equals 0
   Q: y equals 0. The given statement is thus $P \vee Q$.
   (iv) P: $x \in A$; Q: $x \in B$. Therefore, the given statement is $P \wedge Q$.

5. State the negation of each of the following statements without using the word "not."
   (a) The real number r is at most 2.
   (b) The absolute value of the number a is less than 3.
   (c) Two sides of the triangle have the same length.
   (d) No one expected it to rain.
   (e) It is surprising that two students received the same exam score.

Solution:

(a) The real number r is greater than 2.

(b) The absolute value of the number a is at least 3.

(c) The sides of the triangle have different lengths.

(d) Someone expected it to rain.

(e) It is expected that two students received the same exam score.

6. Consider the statements P : 17 is even and Q : 19 is prime. Write each of the following statements in words and indicate whether it is true or false.

   (a) $\sim P$

   (b) $P \wedge Q$

   (c) $P \vee Q$

   Solution:

   (a) $\sim P$ is "17 is odd." True, because 17 is odd.

   (b) $P \wedge Q$ is "17 is even and 19 is prime." False. Although Q is true, P is false. Therefore, $P \wedge Q$ is false.

   (c) $P \vee Q$ is "17 is even or 19 is prime." True. Although P is false, Q is true. Therefore, $P \vee Q$ is true.

7. Without changing their meanings, convert each of the following sentences into a sentence having the form *"If P, then Q."*

   (a) Whenever three sides of a triangle are equal, the angles of the triangle are equal.

   (b) The square of every integer is positive.

   (c) The integer n^3 is even only if n is even.

   Solution:

   (a) If three sides of a triangle are equal, the angles if the triangle are equal.

   (b) If a number is an integer, then its square is positive.

   (c) If the integer n is even, then n^3 is also even.

8. Without changing their meanings, convert each of the following sentences into a sentence having the form *"P if and only if Q."*

   (a) If a function has constant derivative, it is linear, and conversely.

   (b) For a circle to have both a perimeter and an area of 4π, it is necessary and sufficient that its radius be 2.

   Solution:

   (a) A function has a constant derivative if and only if it is linear.

   (b) A circle has both a perimeter and an area of 4π if and only if its radius is 2.

9. Consider the statements P: $\sqrt{2}$ is rational and Q: $\frac{22}{7}$ is rational. Write each of the following statements in words and indicate whether it is true or false.

   (a) $P \implies Q$

   (b) $Q \implies P$

   (c) $P \iff Q$

   Solution:

   (a) $P \implies Q$ is "If $\sqrt{2}$ is rational, then $\frac{22}{7}$ is rational." True, since $\frac{22}{7}$ is rational.

   (b) $Q \implies P$ is "If $\frac{22}{7}$ is rational, then $\sqrt{2}$ is rational." False, since $\frac{22}{7}$ is rational but $\sqrt{2}$ is not.

   (c) $P \iff Q$ is "$\sqrt{2}$ is rational if and only if $\frac{22}{7}$ is rational." False, since $\sqrt{2}$ is not rational.