Homework # 8 Solutions

Math 111, Fall 2014
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1. Suppose $x \in \mathbb{Z}$. Then x is odd if and only if $3x + 5$ is even.

   Solution:

   Proof.

   $\boxed{\Longrightarrow}$ Let x be an odd integer. Then $x = 2a + 1$ for some integer a. So,

   $$3x + 5 = 3(2a + 1) + 5 = 6a + 8 = 2(3a + 4).$$

   Since $3a + 4$ is an integer, $3x + 5$ is even.

   $\boxed{\Longleftarrow}$ (contrapositive) Suppose that x is an even integer. Then $x = 2b$ for some integer b. So,

   $$3x + 5 = 3(2b) + 5 = 6b + 5 = 2(3b + 2) + 1.$$

   Since $3b + 2$ is an integer, $3x + 5$ is odd.

   □

2. Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2 y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

   Solution:

   Proof.

   $\boxed{\Longrightarrow}$ Let $x, y \in \mathbb{R}$, and suppose that $x^3 + x^2 y = y^2 + xy$. Then we have

   $$x^2(x + y) = y(x + y).$$

   If $x + y \neq 0$, then we may divide both sides by $x + y$ to obtain

   $$x^2 = y.$$

   If $x + y = 0$, then we obtain

   $$0 = 0,$$

   a true statement. But $x + y = 0$ is the same as $y = -x$.

   Therefore, if $x^3 + x^2 y = y^2 + xy$, then $y = x^2$ or $y = -x$.

$\Longleftarrow$ Let $x, y \in \mathbb{R}$. We have two possibilities: either $y = x^2$ or $y = -x$. If $y = x^2$, then

$$x^3 + x^2y = x^3 + x^2(x^2) = x^3 + x^4,$$

and

$$y^2 + xy = (x^2)^2 + x(x^2) = x^4 + x^3 = x^3 + x^2y.$$

If $y = -x$, then

$$x^3 + x^2y = x^3 + x^2(-x) = x^3 - x^3 = 0,$$

and

$$y^2 + xy = (-x)^2 + x(-x) = x^2 - x^2 = 0,$$

and

$$y^2 + xy = (x^2)^2 + x(x^2) = x^4 + x^3 = x^3 + x^2y.$$

Therefore, if $y = x^2$ or $y = -x$, then $x^3 + x^2y = y^2 + xy$.

□

3. Suppose $a, b \in \mathbb{Z}$. Then $a \equiv b \pmod{10}$ if and only if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.

Solution:

Proof.

$\Longrightarrow$ Suppose a and b are integers such that $a \equiv b \pmod{10}$. Then $10 \mid (a-b)$, so there exists an integer k such that $a - b = 10k$. But, this means that $a - b = 2(5k)$, and since $5k$ is an integer, $2 \mid (a - b)$ so $a \equiv b \pmod{2}$. This also means that $a - b = 5(2k)$, and since $2k$ is an integer, $5 \mid (a - b)$, so $a \equiv \pmod{5}$.

$\Longleftarrow$ Suppose a and b are integers such that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. Then, $2 \mid (a - b)$ and $5 \mid (a - b)$. Therefore, there exist integers m and n such that $a - b = 2m$ and $a - b = 5n$. Equating these expressions gives

$$2m = 5n.$$

Since the left-hand side is $2m$, an even integer, we know that $5n$ must be even. Since 5 is not even, we know that n must be even. Therefore, $n = 2a$ for some integer a, and we obtain

$$2m = 5(2a) = 10a.$$

So, we have that $a - b = 10a$, which means that $10 \mid (a - b)$, or $a \equiv b \pmod{10}$.

□

4. There exists a positve real number x for which $x^2 < \sqrt{x}$.

   Solution: Let $x = \dfrac{1}{4}$. Then, since $\sqrt{x} = \dfrac{1}{2}$ and $x^2 = \dfrac{1}{16} < \sqrt{x}$, $x = \dfrac{1}{4}$. is one such positive real number.

5. There is a set X such that $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

   Solution: Let $X = \mathbb{N} \cup \{\mathbb{N}\}$. Then $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$, so $X = \mathbb{N} \cup \{\mathbb{N}\}$ is one such set.

6. Suppose $a, b \in \mathbb{N}$. Then $a = \operatorname{lcm}(a, b)$ if and only if $b \mid a$.

   Solution:

   Proof.

   $\Longrightarrow$ Let a and b be two natural numbers such that $a = \operatorname{lcm}(a, b)$. Then there exist integers m and n such that $am = a$ and $bn = a$. Therefore, $a \mid a$ (obvious) and $b \mid a$.

   $\Longleftarrow$ (direct) Suppose that $b \mid a$. Then $a = bk$ for some integer k. So,

   $$\operatorname{lcm}(a, b) = \operatorname{lcm}(bk, b) = bk = a.$$

   □

   The converse (If $b \mid a$, then $a = \operatorname{lcm}(a, b)$ could also be proven by contradiction, as follows.

   Proof. (contradiction) Suppose to the contrary that $b \mid a$ and $a \neq \operatorname{lcm}(a, b)$. Since $a \neq \operatorname{lcm}(a, b)$ and $a \mid a$, it must be true that $b \nmid a$. This contradicts our assumption that $b \mid a$. □

7. For every real number x, there exist integers a and b such that $a \leq x \leq b$ and $b - a = 1$.

   Solution: Let $x \in \mathbb{R}$. There are two possibilities: x is an integer or x is not an integer. If x is an integer, then let $a = x$ and $b = x + 1$. Then $a \leq x \leq b$ and $b - a = 1$. If x is not an integer, then we can find an integer a such that $a < x < a + 1$. So, let $b = a + 1$. Then $a \leq x \leq b$ and $b - a = 1$.