Proof of Second Proposition from Class on October 20, 2014

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Proposition. Let $x, y \in \mathbb{Z}$. Then $4 \mid (x^2 - y^2)$ if and only if x and y are of the same parity.

Proof.

$\Longrightarrow$ We will do a proof by contrapositive. Assume that x and y have opposite parity. Without loss of generality, assume that x is even and y is odd. Then $x = 2k$ and $y = 2l + 1$ for some integers k and l. Then, we have

$$\begin{aligned} x^2 - y^2 &= (2k)^2 - (2l+1)^2 \\ &= 4k^2 - (4l^2 + 4l + 1) \\ &= 4k^2 - 4l^2 - 4l - 1 \\ &= 4k^2 - 4l^2 - 4l - 4 + 3 \\ &= 4(k^2 - l^2 - l - 1) + 3. \end{aligned}$$

Since $k^2 - l^2 - l - 1$ is an integer, it follows that there is a remainder of 3 when $x^2 - y^2$ is divided by 4. Therefore, $4 \nmid (x^2 - y^2)$.

$\Longleftarrow$ Assume that x and y have the same parity. We want to show that $4 \mid (x^2 - y^2)$. There are two cases.

Case 1: Both x and y are even. Then $x = 2a$ and $y = 2b$ for some integers a and b. Then,

$$x^2 - y^2 = (2a)^2 - (2b)^2 = 4a^2 - 4b^2 = 4(a^2 - b^2).$$

Since $a^2 - b^2$ is an integer, $4 \mid (x^2 - y^2)$.

Case 2: Both x and y are odd. Then, $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n. So,

$$\begin{aligned} x^2 - y^2 &= (2m+1)^2 - (2n+1)^2 \\ &= (4m^2 + 4m + 1) - (4n^2 + 4n + 1) \\ &= 4m^2 + 4m - 4n^2 - 4n \\ &= 4(m^2 + m - n^2 - n). \end{aligned}$$

Since $m^2 + m - n^2 - n$ is an integer, $4 \mid (x^2 - y^2)$.

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