

Fundamentals – Part 1 of Hammack

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1 Sets

1.1 Introduction to Sets

A **set** is a collection of things called **elements**. Sets are sometimes represented by a comma-separated list of elements surrounded by curly braces. To indicate that a set contains infinitely many elements, we can write it as follows:

$$\{\dots, -3, -1, 1, 3, 5, \dots\}$$

is the set of all odd integers. This is an example of an **infinite** set; a **finite** set contains finitely many elements. Two sets are said to be **equal** if they contain the same elements, although not necessarily in the same order. For example,

$$\{1, 3, 5, 7\} = \{7, 5, 3, 1\},$$

but

$$\{\dots, -4, -2, 0, 2, 4, \dots\} \neq \{\dots, -3, -1, 1, 3, 5, \dots\}.$$

Typically, capital letters are used to refer to sets. For example, we could say

$$O = \{\dots, -3, -1, 1, 3, 5, \dots\}.$$

Then, to say that -3 is in O , we would write $-3 \in O$. To say 4 is not in O , we would write $4 \notin O$. There are some sets that you already know. For example,

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$;
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$;
- \mathbb{R} is the set of all real numbers.

Sets can contain things other than numbers. For example, sets can contain letters. The set $B = \{T, F\}$ and the set $S = \{\sin t, \cos t, \sin^2 t, \cos^2 t\}$ are two examples. In fact, a set can contain other sets. For example, the set $E = \{1, \{1, 2\}, \{1, 2, 3\}\}$ contains the element 1 and two sets, $\{1, 2\}$ and $\{1, 2, 3\}$. We note that $1 \in E$ and $\{1, 2\} \in E$ and $\{1, 2, 3\} \in E$, but $2 \notin E$ and $3 \notin E$.

The number of elements in a finite set is called its **cardinality** (or **size**) and is denoted by absolute value. For example, the cardinality of set B above is $|B| = 2$. What is $|E|$?

Another important set is the **empty set**, \emptyset . The empty set is the set that has no elements, so $\emptyset = \{\}$. It makes sense, then, that $|\emptyset| = 0$.

Finally, let's talk about creating our own sets using a notation (called set-builder notation in the text) that is simpler than just listing all of the elements. The requisite ingredients are

- the form of the set (an expression);
- a colon, which denotes "such that"; and
- a rule.

For example, the set O as defined above may be written as

$$O = \{2n + 1 : n \in \mathbb{Z}\}.$$

This means that O is the set of all numbers of the form $2n + 1$ such that n is an element of \mathbb{Z} . In other words, O is the set of all numbers of the form $2n + 1$, where n is an integer. Another way to write this set is

$$O = \{n \in \mathbb{Z} : n \text{ is odd}\},$$

read O is the set of all integers n such that n is odd. Note that some authors use notation

$$X = \{\text{expression}|\text{rule}\},$$

where the $|$ replaces the colon.

Here are some other examples:

- $\{x \in \mathbb{N} : x < 3\} = \{1, 2\}$
- $\{x^2 + y^2 : x, y \in \mathbb{Z}\}$
- $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 1\}$

One other important set is the set of **rational numbers**, which can be written as

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

Also, we may define the set of complex numbers as

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}.$$

Exercise: Write in set-builder notation

(1) $\{2, 4, 8, 16, 32\}$

(2) $\{\dots, -6, -3, 0, 3, 6, 9, \dots\}$

You should review interval notation from Calculus, as well.

1.2 The Cartesian Product

Given two sets, A and B , we may form their Cartesian product, denoted $A \times B$. What does this mean?

Definition. An **ordered pair** is a list (x, y) of two elements x and y enclosed in parentheses and separated by commas.

Anything in parentheses is an ordered pair. For example (a, e) is an ordered pair, as is $(\{-1, 2\}, \{-1, 0\})$, which is an ordered pair of sets. You can even have an ordered pair of ordered pairs, like $((x, y), (z, w))$. Note that for ordered pairs, order counts; i.e., $(a, b) \neq (b, a)$.

Definition. The **Cartesian product** of two sets A and B is another set denoted $A \times B$ and defined as

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example: Suppose $A = \{1, 4, 7, 10\}$ and $B = \{2, 5, 8\}$. Then

$$A \times B = \{(1, 2), (1, 5), (1, 8), (4, 2), (4, 5), (4, 8), (7, 2), (7, 5), (7, 8), (10, 2), (10, 5), (10, 8)\}.$$

We can easily show that if A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

We can define Cartesian products of Cartesian products, such as $\mathbb{R} \times (\mathbb{N} \times \mathbb{Z}) = \{(x, (y, z)) : x \in \mathbb{R}, (y, z) \in \mathbb{N} \times \mathbb{Z}\}$.

We can also define Cartesian products of three or more sets by moving beyond ordered pairs. For example, an **ordered triple** is a list (x, y, z) . The Cartesian product of the three sets \mathbb{R} , \mathbb{N} , and \mathbb{Z} is $\mathbb{R} \times \mathbb{N} \times \mathbb{Z} = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{N}, z \in \mathbb{Z}\}$. The Cartesian product $\mathbb{R} \times (\mathbb{N} \times \mathbb{Z})$ is different from the Cartesian product $\mathbb{R} \times \mathbb{N} \times \mathbb{Z}$: the former is a Cartesian product of two sets, and the latter is the Cartesian product of three sets.

Finally, we may take **Cartesian powers** of sets. For any set A and positive integer n , the power A^n is the Cartesian product of A with itself n times. For example,

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

We may also define integer lattices, such as

$$\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

Exercise: Suppose $A = (\pi, e, 0)$ and $B = (0, 1)$. Write the following sets by listing their elements between braces.

- (1) $A \times B$
- (2) $B \times B$
- (3) $A \times (B \times B)$
- (4) $A \times B \times B$

1.3 Subsets

The elements in one set may also be elements of another set. One example of this is \mathbb{N} : all of the numbers in \mathbb{N} are also in \mathbb{Z} . When two sets A and B are related this way, we say that A is a subset of B .

Definition. Suppose that A and B are sets. If every element of A is also an element of B , we say that A is a **subset** of B , and we denote this as $A \subseteq B$. We write $A \not\subseteq B$ if A is *not* a subset of B ; i.e., if it is *not* true that every element of A is also an element of B . Thus, $A \not\subseteq B$ means that there is at least one element of A that is *not* an element of B .

Example: Determine whether the following are true or false, and explain your answer.

- (1) $\{0, 1, 2\} \subseteq \{0, 1, 2, 4, 8\}$
- (2) $\{0, 2, 3\} \subseteq \{0, 1, 2, 4, 8\}$
- (3) $\{2n + 1 : n \in \mathbb{Z}\} \subseteq \mathbb{Z}$
- (4) $\{m^2 + n^2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{N}$

Solution:

- (1) $\{0, 1, 2\} \subseteq \{0, 1, 2, 4, 8\}$ - True, since the elements of the first set are contained in the larger set.

- (2) $\{0, 2, 3\} \subseteq \{0, 1, 2, 4, 8\}$ - False, because $3 \notin \{0, 1, 2, 4, 8\}$.
- (3) $\{2n + 1 : n \in \mathbb{Z}\} \subseteq \mathbb{Z}$ - True, because the first set consists of odd integers, which are clearly contained in the set of all integers.
- (4) $\{m^2 + n^2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{N}$ - False. If $m = n = 0$, then $m^2 + n^2 = 0 \notin \mathbb{N}$.

Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

It should be clear that the empty set is a subset of every set; i.e., $\emptyset \subseteq B$ for any set B .

Next, we will work on enumerating the subsets of a finite set.

Question: How many elements will we get?

Answer: We get 2^n subsets, where n is the number of elements in the set.

Example: Let $A = \{1, 2, 3, \{4, 5\}\}$. List all of the subsets of A .

Solution: $\emptyset, \{1\}, \{2\}, \{3\}, \{\{4, 5\}\}, \{1, 2\}, \{1, 3\}, \{1, \{4, 5\}\}, \{2, 3\}, \{2, \{4, 5\}\}, \{3, \{4, 5\}\}, \{1, 2, \{4, 5\}\}, \{1, 3, \{4, 5\}\}, \{2, 3, \{4, 5\}\}, \{1, 2, 3\}, \{1, 2, 3, \{4, 5\}\}$.

You can see from this that the set $\{3, 5\}$ is not a subset of A , because 5 is not an element of A .

Example: $\{1, 2\} \not\subseteq \{\{1, 2\}\}$, but $\{1, 2\} \in \{\{1, 2\}\}$. The only subsets of $\{\{1, 2\}\}$ are \emptyset and $\{\{1, 2\}\}$.

Exercise: List all of the subsets of the following sets.

- (1) $\{1, 2, \emptyset\}$
- (2) $\{\emptyset\}$
- (3) $\{\mathbb{Z}\}$

Do you obtain the expected number of subsets?

Solution:

- (1) $\{1, 2, \emptyset\} = \emptyset, \{\emptyset\}, \{1\}, \{2\}, \{1, 2\}, \{1, 2, \emptyset\}$
- (2) $\{\emptyset\} = \emptyset, \{\emptyset\}$
- (3) $\{\mathbb{Z}\} = \emptyset, \{\mathbb{Z}\}$

Examples of more commonly encountered subsets are

- $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- $G = \{(x, f(x)) : x \in \mathbb{R}\}$ (the graph of $f(x)$). Note that $G \subseteq \mathbb{R}^2$.

Exercise: Determine if the following statements are true or false. Explain your answer.

- (1) $\mathbb{R}^2 \subseteq \mathbb{R}^3$
- (2) $\{(x, y) : x - 1 = 0\} \subseteq \{(x, y) : x^2 - x = 0\}$

Solution: (1) is false, and (2) is true. Why?

1.4 Power Sets

Definition. If A is a set, the **power set** of A is another set, denoted as $\mathcal{P}(A)$ and defined to be the set of all subsets of A . Symbolically, $\mathcal{P}(A) = \{X : X \subseteq A\}$.

We have done some examples of finding all of the subsets of a finite set. Let's do a shorter example. Let $A = \{a, b, c\}$. Then the subsets of A are $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Therefore,

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

We also noted that if a finite set A contains n elements, then it has 2^n subsets, and thus, its power set has 2^n elements. So, we have

$$|\mathcal{P}(A)| = 2^{|A|}.$$

Examples:

- (1) $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{\{1, 2\}\}\}$
- (2) $\mathcal{P}(1)$ is undefined
- (3) $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$
- (4) $\mathcal{P}(\{\{1, 2\}\}) = \{\emptyset, \{\{1, 2\}\}\}$
- (5) $\mathcal{P}(\{1, \{1, 2\}\}) = \{\emptyset, 1, \{\{1, 2\}\}, \{1, \{1, 2\}\}\}$

If A is infinite, clearly we cannot write $\mathcal{P}(A)$ as we did above, because $\mathcal{P}(A)$ contains infinitely many elements. For example, $\mathcal{P}(\mathbb{Z})$ contains all subsets of \mathbb{Z} , but since \mathbb{Z} is the set of all integers, $\mathcal{P}(\mathbb{Z})$ consists of all subsets consisting of some number of integers, and the empty set.

Exercise: Find the following sets.

- (1) $\mathcal{P}(\{\{a\}, \{b, c\}\})$
- (2) $\mathcal{P}(\{\{\emptyset\}, \pi\})$
- (3) $\mathcal{P}(\mathcal{P}(\{2\}))$
- (4) $\mathcal{P}(\{a, b\} \times \{0\})$

Solution:

- (1) $\mathcal{P}(\{\{a\}, \{b, c\}\}) = \{\emptyset, \{\{a\}\}, \{\{b, c\}\}, \{\{a\}, \{b, c\}\}\}$
- (2) $\mathcal{P}(\{\{\emptyset\}, \pi\}) = \{\emptyset, \{\{\emptyset\}\}, \{\pi\}, \{\{\emptyset\}, \pi\}\}$
- (3) $\mathcal{P}(\{2\}) = \{\emptyset, \{2\}\} \implies \mathcal{P}(\mathcal{P}(\{2\})) = \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}$
- (4) $\mathcal{P}(\{a, b\} \times \{0\}) = \{\emptyset, \{(a, 0)\}, \{(b, 0)\}, \{(a, 0), (b, 0)\}\}$

Exercise: Suppose that $|A| = a$ and $|B| = b$. Determine the following.

- (a) $|\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))|$
- (b) $|\{X \in \mathcal{P}(A) : |X| \leq 1\}|$
- (c) $|\mathcal{P}(A \times B)|$

Solution:

- (a) $|\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))| = 2^{(2^{(2^a)})}$
- (b) $|\{X \in \mathcal{P}(A) : |X| \leq 1\}| = a + 1$
- (c) $|\mathcal{P}(A \times B)| = 2^{ab}$

1.5 Union, Intersection, Difference

Union, intersection, and difference are three additional operations that can be applied to sets.

Definition. Suppose A and B are sets.

The **union** of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

The **intersection** of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

The **difference** of A and B is the set $A - B = \{x : x \in A \text{ and } x \notin B\}$.

A few rules: Suppose that A and B are sets. Then the following are true (verify).

- $A \cup B = B \cup A$;
- $A \cap B = B \cap A$; and
- $A - B \neq B - A$ in general.

Example: Suppose that $A = \{1, 2, 3, 4, 5\}$, $B = \{6, 4, 2\}$, and $C = \{a, b, c\}$. Find:

- (a) $A \cup B$
- (b) $A \cap B$
- (c) $A - B$
- (d) $B - A$
- (e) $A \cap C$
- (f) $(A \cap B) \cup C$
- (g) $(A \cup B) \cap (A \cup C)$
- (h) $\mathcal{P}(A - B)$

Solution:

- (a) $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
- (b) $A \cap B = \{2, 4\}$
- (c) $A - B = \{1, 3, 5\}$
- (d) $B - A = \{6\}$
- (e) $A \cap C = \emptyset$
- (f) $(A \cap B) \cup C = \{2, 4, a, b, c\}$
- (g) $(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 5\}$
- (h) $\mathcal{P}(A - B) = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$

Exercise: Suppose that $A = \{0, 1\}$ and $B = \{1, 2\}$. Find:

- (a) $(A \cap B) \times A$

- (b) $(A \times B) \cap B$
- (c) $\mathcal{P}(A) - \mathcal{P}(B)$
- (d) $\mathcal{P}(A \cap B)$

Solution:

- (a) $(A \cap B) \times A = \{(1, 0), (1, 1)\}$
- (b) $(A \times B) \cap B = \emptyset$
- (c) $\mathcal{P}(A) - \mathcal{P}(B) = \{\{0\}, \{0, 1\}\}$
- (d) $\mathcal{P}(A \cap B) = \{\emptyset, \{1\}\}$

1.6 Complement

To discuss the operation of set complement, we first need to discuss the idea of a universal set. When we are talking about a set, we almost always regard it as a subset of some larger set. For example, the set $A = \{1, 3, 5\}$ is a subset of \mathbb{N} and the set $B = \{\pi, e, \sqrt{2}\}$ is a subset of \mathbb{R} . These larger sets are called the **universal sets** or **set universes** for A and B , respectively. Almost every set that we will use can be regarded as having some natural universal set. For example, the set $G = \{(x, f(x)) : x \in \mathbb{R}\}$ (the graph of a function $f(x)$) consists of points in the plane \mathbb{R}^2 , so it is natural to regard \mathbb{R}^2 as the universal set for G .

If we don't have a specific set, the universal set is often denoted U .

Definition. Let A be a set with a universal set U . The **complement** of A , denoted \bar{A} , is the set $\bar{A} = U - A$.

Example: $O = \{2n - 1 : n \in \mathbb{N}\}$ has as its complement all of the even natural numbers, so

$$\bar{O} = \mathbb{N} - O = \{2n : n \in \mathbb{N}\}.$$

Example: Let $X = [0, 1] \times [1, 2]$. Sketch the set \bar{X} and the set $\bar{X} \cap ([-2, 3] \times [-1, 4])$.

Solution: $[0, 1] \times [1, 2]$ is the rectangle (including the boundary) whose vertices are at $(0, 1)$, $(0, 2)$, $(1, 1)$, and $(1, 2)$. So, \bar{X} is all of \mathbb{R}^2 except for the rectangle and its boundary. Then, $\bar{X} \cap ([-2, 3] \times [-1, 4])$ is the portion of $[-2, 3] \times [-1, 4]$ which is outside the boundary of the rectangle.

Exercise: Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n : n \in \mathbb{Z}\}$. The universal set for A and B is then \mathbb{Z} . Find: (a) \bar{A} ; (b) \bar{B} ; (c) $A \cap \bar{A}$; (d) $A \cup \bar{A}$; (e) $\overline{\bar{A} \cup \bar{B}}$

Solution:

- (a) $\bar{A} = B$
- (b) $\bar{B} = A$
- (c) $A \cap \bar{A} = \emptyset$
- (d) $A \cup \bar{A} = \mathbb{Z}$
- (e) $\overline{A \cup B} = \emptyset$

1.7 Venn Diagrams

When working with sets, it is sometimes helpful to draw informal schematic diagrams of them. When we do this, we often represent a set with a circle (or oval), which we consider as containing all of the elements of the set. These diagrams can illustrate the results of various operations on sets, such as union, intersection, and difference. These representations of sets are called **Venn diagrams**.

Example: Given a set A with universal set U , sketch the Venn diagram for \bar{A} .

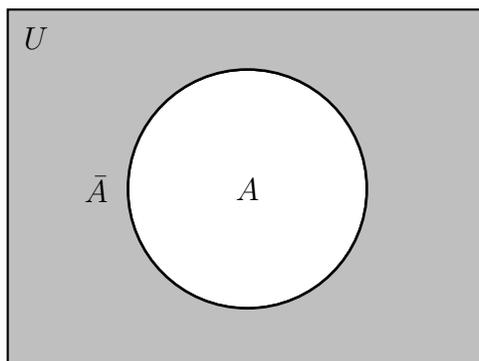


Figure 1: Venn diagram for \bar{A} .

Figure 2 shows some Venn diagrams illustrating operations with two sets, A and B .

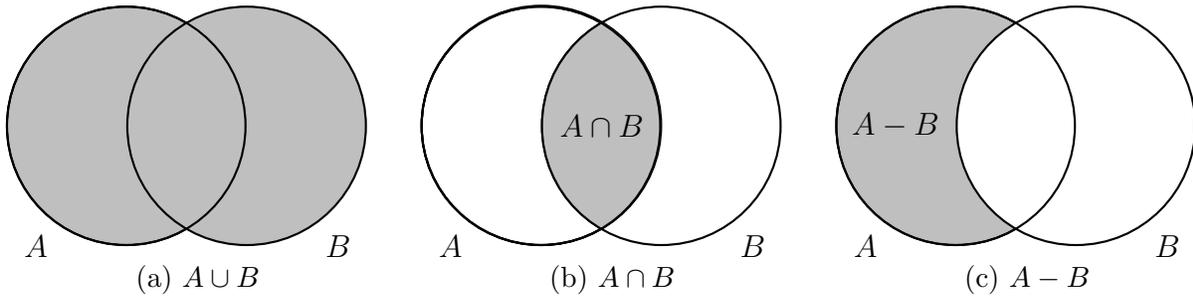


Figure 2: Venn diagrams for two sets.

Figure 3 shows two Venn diagrams for three sets, A , B , and C .

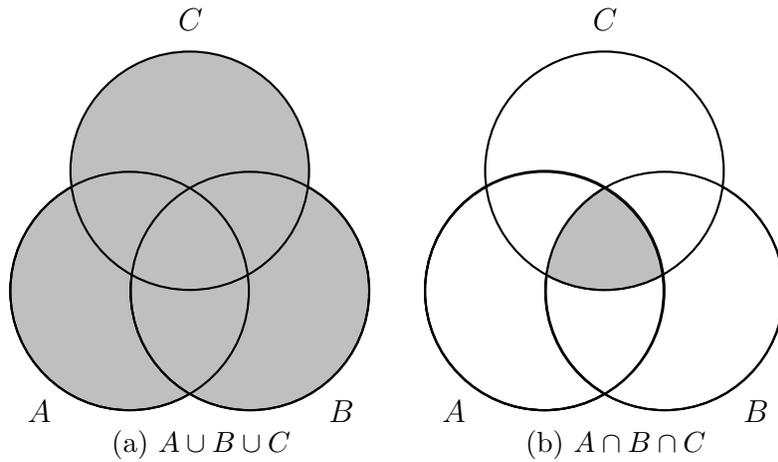


Figure 3: Venn diagrams for three sets.

Example: Sketch the Venn diagram for $(A - B) \cap C$. In the diagram for the intermediate step, $A - B$ is denoted by the horizontal hash lines and C is shaded in by the vertical hash lines. The intersection of the two regions is the cross-hatched area in Figure 4(a), and the final solution is sketched in Figure 4(b).

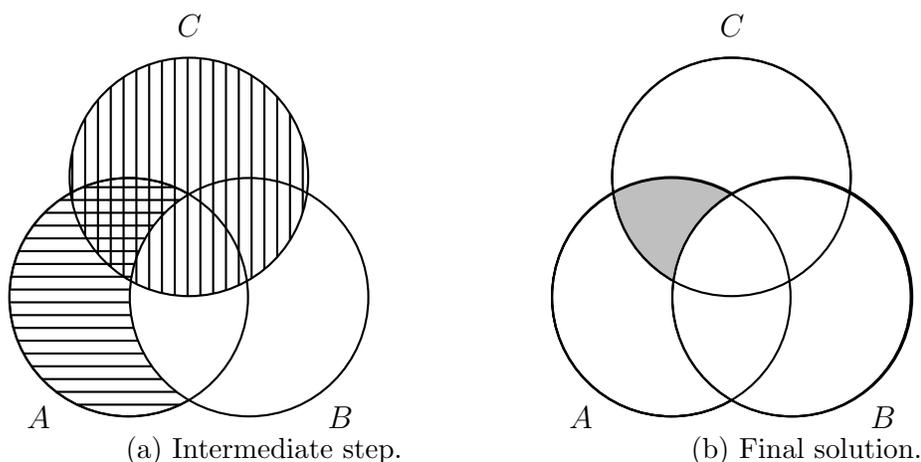


Figure 4: Drawing Venn diagrams for three set example.

Exercise: Sketch the Venn diagram for $A - (B \cap C)$.

1.8 Indexed Sets

Often, mathematical problems involve many sets. In this case, it is convenient to label them A_1, A_2, A_3 , etc. These are called **indexed sets**.

We can use indexed sets to define set operations for any number of sets.

Definition. Suppose A_1, A_2, \dots, A_n are sets. Then

$$A_1 \cup A_2 \cup \dots \cup A_n = \{x : x \in A_i \text{ for at least one set } A_i, \text{ for } 1 \leq i \leq n\};$$

$$A_1 \cap A_2 \cap \dots \cap A_n = \{x : x \in A_i \text{ for every set } A_i, \text{ for } 1 \leq i \leq n\}.$$

Another way to write this is to use notation similar to sigma, or summation, notation that you saw in Calculus. In other words, we write

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n \text{ and } \bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n.$$

We can use this notation even if we have infinitely many sets:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \dots \text{ and } \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \dots$$

Then,

$$\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for at least one set } A_i \text{ with } 1 \leq i\};$$
$$\bigcap_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for every set } A_i \text{ with } 1 \leq i\}.$$

Example: Let $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, \dots , $A_n = \{0, 1, 2, \dots, n\}$, \dots . Find

(a) $\bigcup_{i=1}^{\infty} A_i$

(b) $\bigcap_{i=1}^{\infty} A_i$

Solution:

(a) $\bigcup_{i=1}^{\infty} A_i = \{0, 1, 2, \dots, n\} = \{0\} \cup \mathbb{N}$

(b) $\bigcap_{i=1}^{\infty} A_i = \{0, 1\}$

One other way we can use this notation is to choose indices from a set. For example,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i.$$

In general, the way this works is that we have a collection of sets for $i \in I$, where the set I is called the index set. The set I does not need to consist of integers, but could be anything, including letters, real numbers, etc. We will make a change of notation, indexing the sets using α , instead of i , so as to avoid confusion.

Definition. If we have a set A_α for every α in some index set I , then

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for at least one set } A_\alpha \text{ with } \alpha \in I\};$$
$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for every set } A_\alpha \text{ with } \alpha \in I\}.$$

Example: Consider the sets $A_\alpha = \{\alpha\}$, where $\alpha \in \mathbb{R}$. Find

$$(a) \bigcup_{\alpha \in \mathbb{R}} A_\alpha$$

$$(b) \bigcup_{\alpha \in \mathbb{R}} A_\alpha \times [0, 1]$$

$$(c) \bigcap_{\alpha \in \mathbb{R}} A_\alpha$$

Solution:

$$(a) \bigcup_{\alpha \in \mathbb{R}} A_\alpha = \mathbb{R}$$

$$(b) \bigcup_{\alpha \in \mathbb{R}} A_\alpha \times [0, 1] = \{(x, y) : x \in \mathbb{R}, 0 \leq y \leq 1\}$$

$$(c) \bigcap_{\alpha \in \mathbb{R}} A_\alpha = \emptyset$$

Exercises: Find the following.

$$(1) \bigcup_{i \in \mathbb{N}} [i, i + 1]$$

$$(2) \bigcap_{i \in \mathbb{N}} [0, i + 1]$$

$$(3) \bigcap_{X \in \mathcal{P}(\mathbb{N})} X$$

Solution:

$$(1) \bigcup_{i \in \mathbb{N}} [i, i + 1] = [1, \infty)$$

$$(2) \bigcap_{i \in \mathbb{N}} [0, i + 1] = [0, 2]$$

$$(3) \bigcap_{X \in \mathcal{P}(\mathbb{N})} X = \emptyset$$

1.9 Sets that Are Number Systems

For this section of the text, we are most interested in two facts. First, we assume that the sets in which we are most interested satisfy the **well-ordering principle**, which means that we can put the entries in the set into some order and that every non-empty subset has a smallest element.

The second fact is given by the **division algorithm**:

The Division Algorithm. Given integers a and b with $b > 0$, there exist integers q and r for which $a = qb + r$ and $0 \leq r < b$.

See the text for more details.

2 Logic

Logic is a systematic way of thinking that allows us to deduce new information from known information and to parse the meaning of sentences. In this chapter, we discuss logic in a systematic way.

2.1 Statements

In logic, a **statement** is a sentence or mathematical expression that is either definitely true or definitely false.

Examples of true statements:

- This class meets from 3:30-4:45 p.m. on Mondays and Wednesdays.
- 3 is an odd integer.
- $\pi \in \mathbb{R}$.

Examples of false statements:

- $\sqrt{2}$ is an integer.
- $-3 \in \mathbb{N}$.

The textbook will often use the letters $P, Q, R,$ and S to stand for specific statements, as well as these letters with subscripts. For example:

P : For every integer $n > 1$, $2^n > 2$.

Q_1 : Every $x \in \mathbb{N}$ is also an integer.

Q_2 : Every rational number is also a real number.

Statements may also contain variables. If so, then the statement may be denoted like a function, e.g.,

$P(x)$: If x is an integer that is divisible by 4, then x is also divisible by 2.

An **open sentence** is a sentence whose truth depends on the value of one or more variables. For example,

P : $f(x)$ is continuous on $[0, \infty)$.

Conjectures are open sentences for which it is not yet known whether the sentence is true or false. Two of the most famous open sentences are Fermat's last theorem (which was fairly recently proved and is technically no longer an open sentence but a statement) and the Goldbach conjecture (which says that every integer greater than 2 is the sum of two prime numbers).

Exercise: Determine if the following are statements or open sentences. If the sentence is a statement, determine if it is true or false.

- (1) Every real number is an even integer.
- (2) Sets \mathbb{Z} and \mathbb{N} are infinite.
- (3) $\cos(x) = -1$.
- (4) Either x is a multiple of 7, or it is not.

2.2 And, Or, Not

“And”, “or”, and “not” are three logical operations on statements. Two statements may be combined to make a third, new statement with “and” and “or.”

Let P and Q be two statements. The logical operation “and” is denoted by the symbol \wedge . The statement “ P **and** Q ” (or, equivalently $P \wedge Q$) is true only if both P and Q are true; otherwise, it is false. We can summarize this in a table called a **truth table**, letting “T” represent true and “F” represent false.

Examples:

R_1 : The number 1 is an integer **and** the number $\sqrt{2}$ is a real number. – True.

R_2 : The number -1 is an integer **and** the number $\sqrt{2}$ is an integer. – False.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

R_3 : The number -1 is a natural number **and** the number $\sqrt{2}$ is a real number. – False.

The logical operation “or” is denoted by the symbol \vee . The statement “ P **or** Q ” (or, equivalently $P \vee Q$) is true if at least one of P and Q is true; otherwise, it is false. The truth table for the logical “or” is given below.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples:

S_1 : The number 1 is an integer **or** the number $\sqrt{2}$ is a real number. – True.

S_2 : The number -1 is an integer **or** the number $\sqrt{2}$ is an integer. – True.

S_3 : The number -1 is a natural number **or** the number $\sqrt{2}$ is a real number. – True.

Finally, the logical operator “not” indicates the negation of the statement. For example, the statement “ $\sqrt{2}$ is an integer” can be negated by “It is not true that $\sqrt{2}$ is an integer,” an obviously true statement. The symbol for “not” is \sim , so $\sim P$ means **it is not true that P** . The truth table for $\sim P$ is given below. The statement $\sim P$ is also called the **negation** of P .

P	$\sim P$
T	F
F	T

2.3 Conditional Statements

Another way to combine two statements is using the conditional “If P , then Q ”. This is often written $P \implies Q$, or P implies Q . An example of this is the statement

If the radius of the circle is a real number r , then the circumference of the circle is $2\pi r$.

In this statement, P is the statement “The radius of the circle is a real number r ,” and Q is the statement “The circumference of the circle is $2\pi r$.” We can think of $P \implies Q$ as saying that whenever P is true, then Q will be true also. The only way this will not be true is if P is true, but Q is false. The truth table for $P \implies Q$ is as follows.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

See p. 41 of the textbook for other ways that $P \implies Q$ may be represented in English statements.

2.4 Biconditional Statements

Note that $P \implies Q$ does **not** mean the same thing as $Q \implies P$. For example, consider the following.

P : The radius of the circle is a real number r .

Q : The circumference of the circle is $2\pi r$.

Then, $P \implies Q$ is the statement: If the radius of the circle is a real number r , then the circumference of the circle is $2\pi r$.

The statement $Q \implies P$ is the statement: If the circumference of the circle is $2\pi r$, then the radius of the circle is a real number r .

The conditional statement $Q \implies P$ is called the **converse** of $P \implies Q$.

For the above statements P and Q , we can see that both $P \implies Q$ and $Q \implies P$ are true. Therefore, $(P \implies Q) \wedge (Q \implies P)$ is true. The symbol that expresses this **biconditional** is \iff . The expression $P \iff Q$ means $P \implies Q$ **and** $Q \implies P$, and it is read as P if and only if Q . The truth table for the biconditional is given below.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Why is this true? The first line, where both P and Q are true should be obvious. If one of P or Q is false, then one of $P \implies Q$ or $Q \implies P$ is false, and, therefore, $P \iff Q$ is false. If both P and Q are false, then $P \implies Q$ and $Q \implies P$ are both true, and, so, $P \iff Q$ is true.

2.5 Truth Tables for Statements

You should make sure that you know the truth tables for $\wedge, \vee, \sim, \implies$, and \iff and that you understand these logical operations thoroughly, because we are now going to combine them to make more complex statements. For example, suppose we wish to convey that if at least one of Q or R is true, then both of them are true. Then, we have

$$(P \vee Q) \implies (P \wedge Q).$$

To determine the truth values for this statement, we first determine all possible truth values of $P \vee Q$ and $P \wedge Q$ and then use these values.

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \implies (P \wedge Q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Example: Make a truth table for the statement “If both x and y are positive, then $xy > 0$.”

Solution: First, define the following statements.

P : $x > 0$

Q : $y > 0$

R : $xy > 0$

Then the statement for which we wish to make a truth table is represented by $(P \wedge Q) \implies R$.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \implies R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Exercise: Make a truth table for the statement $\sim P \implies (Q \vee R)$.

Solution:

P	Q	R	$\sim P$	$Q \vee R$	$\sim P \implies (Q \vee R)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	F	F

2.6 Logical Equivalence

Two statements are said to be **logically equivalent** if they have identical truth tables. Two important examples may be found in the textbook:

- the logical equivalence of $P \iff Q$ and $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$; and
- the logical equivalence of $P \implies Q$ and $(\sim Q) \implies (\sim P)$ (the contrapositive law).

To show that two statements are logically equivalent, we use the standard equal sign ($=$). Two pairs of logically equivalent statements that arise quite often are given by **DeMorgan's laws**, which are

$$(1) \quad \sim (P \wedge Q) = (\sim P) \vee (\sim Q)$$

$$(2) \quad \sim (P \vee Q) = (\sim P) \wedge (\sim Q)$$

The first of DeMorgan's laws is verified in the textbook. We shall verify the second as an in-class exercise.

Other laws are fairly straightforward and are given on p. 48 of the textbook. We will look at two of them now.

Examples: Show that the following statements are logically equivalent.

$$(1) \quad P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \text{ (a distributive law)}$$

$$(2) \quad P \vee (Q \vee R) = (P \vee Q) \vee R \text{ (an associative law)}$$

Solution:

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
(1) T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

P	Q	R	$Q \vee R$	$P \vee (Q \vee R)$	$P \vee Q$	$(P \vee Q) \vee R$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
(2) T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	F	F	F

Exercise: Show that the second DeMorgan's law ($\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$) is logically equivalent.

Solution:

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

2.7 Quantifiers

The idea in this section is to express English sentences in symbolic form (and *vice versa*). First, we need two more symbols, called **quantifiers**. These symbols are

- (a) \forall , which means “for all” or “for every” – the **universal quantifier**; and
- (b) \exists , which means “there exists” or “there is” – the **existential quantifier**.

Example: Write the phrase in symbols “for all $\epsilon > 0$, there exists $\delta > 0$ ”

Solution: $\forall \epsilon > 0, \exists \delta > 0$

For now, let's work on translating some symbolic statements as English statements.

Example: Write the following as English statements. Say whether they are true or false. Explain your answer.

- (1) $\forall x \in \mathbb{R}, x^2 > 0$
- (2) $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, ax = x$
- (3) $\forall X \subseteq \mathbb{N}, \exists n \in \mathbb{Z}, |X| = n$
- (4) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m = n + 5$

Solution:

- (1) For every real number x , x^2 is positive. – False. If $x = 0$, then $x^2 = 0$.
- (2) There exists a real number a for which $ax = x$ for every real number x . – True. The number $a = 1$ makes this statement true.
- (3) For all subsets X of \mathbb{N} , there exists an integer n such that the cardinality of X is n . – False. Consider the set $X = \{1, 3, 5, 7, \dots\}$. This set contains infinitely many natural numbers, so there does not exist any integer n for which $|X| = n$.
- (4) For every integer n , there exists an integer m such that $m = n + 5$. – True.

Exercise: Write the following as English statements. Say whether they are true or false. Explain your answer.

- (1) $\forall n \in \mathbb{N}, \exists X \in \mathcal{P}(\mathbb{N}), |X| < n$
- (2) $\exists n \in \mathbb{N}, \forall X \in \mathcal{P}(\mathbb{N}), |X| < n$
- (3) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n + 5$

Solution:

- (1) For every natural number n , there exists a set X in the power set of the natural numbers with the property that the cardinality of X is less than n . – True. Given n , we simply choose a subset of \mathbb{N} that contains $n - 1$ natural numbers.
- (2) There exists a natural number n such that the cardinality of every set X in the power set of the natural numbers is less than n . – False. We know that $\mathbb{N} \in \mathcal{P}(\mathbb{N})$ and \mathbb{N} is an infinite set, so there is no natural number n for which $|X| < n$.
- (3) There exists an integer m such that $m = n + 5$ for all integers n . – False. Given an integer n , there is only one integer m with the property that $m = n + 5$.

2.8 More on Conditional Statements

In mathematics, whenever $P(x)$ and $Q(x)$ are open sentences concerning elements x in some set S (depending on the context of the sentences), an expression of the form $P(x) \implies Q(x)$ is understood to be the statement $\forall x \in S, P(x) \implies Q(x)$. In other words, if a conditional statement does not have an explicit quantifier, then there is an implied universal quantifier in front of it.

Definition. If P and Q are statements or open sentences, then the statement “If P , then Q ” is a statement. The statement is true if it is impossible for P to be true while Q is false. It is false if there is at least one instance in which P is true, but Q is false.

Example:

- If $x \in \mathbb{R}$, then $x^2 \geq 0$. – True.
- If $x \in \mathbb{R}$, then $x > 0$. – False. $-2 \in \mathbb{R}$, and $-2 < 0$.

2.9 Translating English into Symbolic Logic

In this section, we will focus on writing English sentences as expressions involving logic symbols. The purpose of this work is to make sure that you are attentive to the logic structure of theorems so that you understand exactly what the theorem is saying.

Example: Rolle’s Theorem.

Theorem 1 (Rolle’s Theorem). If f is continuous on the interval $[a, b]$ and differentiable on (a, b) , and $f(a) = f(b)$, then there is a number $c \in (a, b)$ for which $f'(c) = 0$.

A translation of this in symbolic form is

$$((f \text{ cont. on } [a, b]) \wedge (f \text{ diff. on } (a, b))) \implies (\exists c \in (a, b), f'(c) = 0).$$

It turns out that every universally quantified statement can be expressed as a conditional statement. In other words:

Fact. Suppose S is a set and $Q(x)$ is a statement about x for each $x \in S$. The following statements mean the same thing:

$$\forall x \in S, Q(x)$$

$$(x \in S) \implies Q(x)$$

Example: “If x is prime, then \sqrt{x} is not a rational number” may be written in the following two ways:

$$(x \text{ prime}) \implies (\sqrt{x} \notin \mathbb{Q})$$

$$\forall x \text{ prime}, \sqrt{x} \notin \mathbb{Q}$$

Exercise: Translate the following sentences into symbolic logic.

- (1) If f is a polynomial and its degree is greater than 2, then f' is not constant.
- (2) For every positive number ϵ , there exists a positive number δ for which $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.
- (3) There exists a real number a for which $a + x = x$ for every real number x .

Solution:

$$(1) ((f \text{ a polynomial}) \wedge (f \text{ has degree greater than } 2)) \implies (f' \text{ is not constant})$$

$$(2) \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists \delta \in \mathbb{R}, \delta > 0, (|x - a| < \delta) \implies (|f(x) - f(a)| < \epsilon)$$

$$(3) \exists a \in \mathbb{R}, \forall x \in \mathbb{R}, a + x = x$$

2.10 Negating Statements

Given a statement R , the statement $\sim R$ is called the **negation** of R . If R is a complex statement, then frequently, $\sim R$ may be written in a simpler or more useful form. The process of finding this form is called **negating** R . In proving theorems, it is often necessary to negate certain statements. We have already seen one example of this with DeMorgan's Laws, which are

$$\sim (P \wedge Q) = (\sim P) \vee (\sim Q) \tag{1}$$

$$\sim (P \vee Q) = (\sim P) \wedge (\sim Q) \tag{2}$$

and can be viewed as rules telling us how to negate the statements $P \wedge Q$ and $P \vee Q$.

Example: Negate the following sentences.

- (1) R : The numbers x and y are both positive.
- (2) R : The number x is positive, but the number y is not positive.

Solution:

- (1) R means (the number x is positive) \wedge (the number y is positive), which we can think of as $P \wedge Q$. Then $\sim R$ is given by $(\sim P) \vee (\sim Q)$, or
 $\sim R$: The number x is not positive or the number y is not positive.
- (2) R can be written as (the number x is positive) \wedge (the number y is not positive), which we can think of as $P \wedge Q$. Then, again, $\sim R$ is given by $(\sim P) \vee (\sim Q)$, or
The number x is not positive or the number y is positive.

It is often necessary to find the negations of quantified statements. For example, consider $\sim (\forall x \in \mathbb{Z}, P(x))$. In words, we have

It is not the case that $P(x)$ is true for all integers x .

This means that $P(x)$ is false for at least one integer x . Symbolically, this is $\exists x \in \mathbb{Z}, \sim P(x)$.

Now, consider $\sim (\exists x \in \mathbb{Z}, Q(x))$. In words, we have that

It is not the case that $Q(x)$ is true for any integer x ,

or, symbolically, $\forall x \in \mathbb{Z}, \sim Q(x)$.

In general, we have

$$\sim (\forall x \in S, P(x)) = \exists x \in S, \sim P(x) \tag{3}$$

$$\sim (\exists x \in S, Q(x)) = \forall x \in S, \sim Q(x) \tag{4}$$

Example: Negate the following sentences.

- (1) For every prime number p , there is another prime number q with $q > p$.
- (2) For every positive number ϵ , there is a positive number M for which $|f(x)| < M$.

Solution:

- (1) Negation: There is a prime number p such that for every prime number q , $q \leq p$.
- (2) Negation: There is a positive number ϵ such that for every positive number M , $|f(x)| \geq M$.

Finally, when proving theorems, we will sometimes need to negate a conditional statement $P \implies Q$. If we look at $\sim (P \implies Q)$, we see that this means that $P \implies Q$ is false. The only way that $P \implies Q$ is false is if P is true and Q is false, or $P \wedge \sim Q$. So, we have

$$\sim (P \implies Q) = P \wedge \sim Q. \tag{5}$$

You can verify this by using a truth table. (See Exercise 12 of Section 2.6).

Example: Negate the following statement “If x is prime, then \sqrt{x} is not a rational number.”

Solution: This translates into $\forall x \in \mathbb{R}, (x \text{ prime}) \implies (\sqrt{x} \notin \mathbb{Q})$. So, the negation of this is found using Equations (3) and (5) to be

$$\begin{aligned} \sim (\forall x \in \mathbb{R}, (x \text{ prime}) \implies \sqrt{x} \notin \mathbb{Q}) &= \exists x \in \mathbb{R}, \sim ((x \text{ prime}) \implies \sqrt{x} \notin \mathbb{Q}) \\ &= \exists x \in \mathbb{R}, (x \text{ prime}) \wedge \sqrt{x} \in \mathbb{Q}. \end{aligned}$$

This translates to: There is a prime number x such that \sqrt{x} is rational.

Hard Example: Negate the following sentence “For every positive number ϵ , there is a positive number M for which $|f(x) - b| < \epsilon$ whenever $x > M$.”

Solution: First, we will write this sentence in symbolic form. The sentence is $\forall \epsilon \in (0, \infty), \exists M \in (0, \infty), (x > M) \implies (|f(x) - b| < \epsilon)$. Now, work out the negation.

$$\begin{aligned} \sim (\forall \epsilon \in (0, \infty), \exists M \in (0, \infty), (x > M) \implies (|f(x) - b| < \epsilon)) \\ &= \exists \epsilon \in (0, \infty), \sim (\exists M \in (0, \infty), \forall x, (x > M) \implies (|f(x) - b| < \epsilon)) \\ &= \exists \epsilon \in (0, \infty), \forall M \in (0, \infty), \sim (\forall x, (x > M) \implies (|f(x) - b| < \epsilon)) \\ &= \exists \epsilon \in (0, \infty), \forall M \in (0, \infty), \exists x, \sim ((x > M) \implies (|f(x) - b| < \epsilon)) \\ &= \exists \epsilon \in (0, \infty), \forall M \in (0, \infty), \exists x, ((x > M) \wedge \sim (|f(x) - b| < \epsilon)) \\ &= \exists \epsilon \in (0, \infty), \forall M \in (0, \infty), \exists x, ((x > M) \wedge (|f(x) - b| \geq \epsilon)). \end{aligned}$$

This translates to: “There exists a positive number ϵ with the property that for every positive number M , there is a number x for which $x > M$ and $|f(x) - b| \geq \epsilon$.”

Exercise: Negate the sentence “If $\sin(x) < 0$, then it is not the case that $0 \leq x \leq \pi$.”

Solution: In symbolic form, we have $\forall x \in \mathbb{R}, (\sin(x) < 0) \implies \sim (0 \leq x \leq \pi)$. So,

$$\begin{aligned} \sim (\forall x \in \mathbb{R}, (\sin(x) < 0) \implies \sim (0 \leq x \leq \pi)) &= \exists x \in \mathbb{R}, \sim ((\sin(x) < 0) \implies \sim (0 \leq x \leq \pi)) \\ &= \exists x \in \mathbb{R}, (\sin(x) > 0) \wedge \sim (\sim (0 \leq x \leq \pi)) \\ &= \exists x \in \mathbb{R}, (\sin(x) > 0) \wedge (0 \leq x \leq \pi). \end{aligned}$$

This translates to: There exists a number x for which $\sin(x) < 0$ and $0 \leq x \leq \pi$.”

2.11 Logical Inference

Basically, **logical inference** is that given two true statements we can infer that a third statement is true.

Example: If P and Q are both true, then $P \wedge Q$ is true; if $P \wedge Q$ is true, then P is true (and Q is true). Finally, if P is true, then $P \vee Q$ is true, regardless of the truth of Q .

3 Counting

3.1 Counting Lists

A **list** is an ordered sequence of objects. A list is denoted by comma separated list of objects surrounded by parentheses. The objects in the list are called the **entries** in the list. Note that a list has a definite order, so $(2, 3, 5, 8) \neq (2, 5, 3, 8)$. Lists may also have repeated entries, like (a, p, p, l, e) . The number of entries in a list is called its **length**. For example, (a, p, p, l, e) has length 5.

A byte is an important type of list. A byte is simply a list of 0's and 1's of length 8, such as $(0, 0, 1, 1, 0, 0, 0, 1)$.

Note that sets and lists, themselves, may be entries in a list. For example, the list $(1, (0, 1), (0, 0, 1, 1))$ is a list of length 3 whose entries consist of one digit and two lists, one of length 2 and one of length 4; and the list (\mathbb{N}, \mathbb{Z}) is a list of length two whose entries are sets. The **empty list** is a list with no entries, denoted $()$.

Two lists are equal if (1) they have the same length, and (2) they have the same entries in exactly the same order. So, $(0, 0, 1, 1) \neq (1, 0, 0, 0)$.

What is the point of all of this? One often needs to count up the number of possible lists that satisfy a certain condition or property. To do this, we often use the **multiplication principle**, which is given below.

Fact (Multiplication Principle). Suppose in making a list of length n , there are a_1 possible choices for the first entry, a_2 possible choices for the second entry, etc. Then the total number of different lists that may be made this way is $a_1 \cdot a_2 \cdot a_3 \cdots a_n$.

Examples:

- (1) Consider lists made from the letters T, H, E, O, R, Y , with repetition allowed.
 - (a) How many length-4 lists are there?
 - (b) How many length-4 lists are there that begin with T ?
 - (c) How many length-4 lists are there that do not begin with T ?
- (2) Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line-ups are there in which all 5 cards are of the same color (i.e., all black or all red)?

Solution:

- (1) (a) Since repetition is permitted, all 6 letters are available for each entry in the length 4 list, so we have $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1,296$ length-4 lists.

- (b) Since the list must begin with the letter T , we have only one choice for the first entry, and 6 choices for each of the remaining 3 entries, giving $1 \cdot 6 \cdot 6 \cdot 6 = 6^3 = 216$ length-4 lists.
- (c) Since the first letter cannot be T , we have 5 choices for the first entry and 6 choices for each of the remaining 3 entries, giving $5 \cdot 6 \cdot 6 \cdot 6 = 1,080$ length-4 lists.
- (2) The first card of the list can be any one of the 52 cards, so there are 52 choices for the first entry. This card is either black or red. In either case, we have only 25 other cards in the deck of the same color. We have no replacement in this problem, so the number of 5-card line-ups where all cards are the same color is $52 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 15,787,200$.

3.2 Factorials

Definition. If n is a non-negative integer, then the **factorial** of n , denoted $n!$, is the number of non-repetitive lists of length n that can be made from n symbols. Thus, $0! = 1$ and $1! = 1$. If $n > 1$, then $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

Example: How many 5-digit integers are there in which there are no repeated digits and all digits are odd?

Solution: There are 5 odd 1-digit integers. So, we have $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ 5-digit integers in which there are no repeated digits and all digits are odd.

Example: How many 9-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if repetition is not allowed, and all of the odd digits must occur first (on the left) followed by all of the even digits.

Solution: $5!4! = 2880$.

Suppose that we wish to use n symbols to form lists of length k . The number of non-repetitive lists of length k whose entries are chosen from a set of n possible entries is

$$\frac{n!}{(n-k)!}.$$

3.3 Counting Subsets

Now, instead of counting the number of lists that can be made by selecting k entries from a set of n possible entries, we will consider the number of subsets that can be made by selecting k elements from a set with n elements. The major difference, now, is that the order is irrelevant (i.e., $\{a, b\} = \{b, a\}$, so both count as one set). How do we do this?

Definition. If n and k are integers, then $\binom{n}{k}$ denotes the number of subsets that can be made by choosing k elements from a set with n elements. The symbol $\binom{n}{k}$ is read “ n choose k .” Some books write $C(n, k)$ instead of $\binom{n}{k}$. $\binom{n}{k}$ is defined for $n, k \in \mathbb{Z}$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Otherwise, $\binom{n}{k} = 0$.

This definition arises because there are $k!$ sets containing the same elements but in a different order. We know this because, given a set of k elements, we can form $k!$ lists of the elements.

Example: How many positive 10-digit integers contain no zeros and exactly three sixes?

Solution: If we wish to make such a number, we can consider that we have 10 blank spaces, and choose three of these spaces for the sixes. There are $\binom{10}{3} = 120$ ways of doing this. For each of these 120 choices, we can fill in the remaining seven blanks with choices from the digits 1, 2, 3, 4, 5, 7, 8, 9, and there are 8^7 ways to do this. Therefore, there are $120 \cdot 8^7 = 251,658,240$ numbers satisfying this condition.

Example: Find $|\{X \in \mathcal{P}(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}) : |X| < 4\}|$.

Solution:

$$\begin{aligned} |\{X \in \mathcal{P}(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}) : |X| < 4\}| &= \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} \\ &= 1 + 10 + 45 + 120 \\ &= 176. \end{aligned}$$

Exercise: How many 16-digit binary strings contain exactly 7 ones?

Solution: Make such a string by starting with a list 16 blank spots. Choose 7 of the blank spots for ones and put zeros in the other blank spots. There are $\binom{16}{7} = 11,440$ such binary strings.

3.4 Pascal’s Triangle and the Binomial Theorem

Pascal’s triangle is an arrangement of numbers in triangular form based on the formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

To show that the above identity is true, recall that $\binom{n+1}{k}$ represents the number of sets with k elements that can be chosen from a set of $n+1$ elements. A simple example of a set of $n+1$ elements is the set $A = \{0, 1, 2, 3, \dots, n\}$. We can consider two types of sets, those containing 0 and those not containing 0. The number of sets not containing 0 is given by $\binom{n}{k}$, and the number of sets containing 0 may be found by forming a set of $k-1$ elements that don't have 0 and then adding 0, which gives $\binom{n}{k-1}$. Since these are all possible ways to form sets of k elements from a set of $n+1$ elements, the identity is proved.

Then, we have (written in the form of a triangle):

$$\begin{array}{cccccccc}
 & & & & & & & \binom{0}{0} \\
 & & & & & & \binom{1}{0} & \binom{1}{1} \\
 & & & & & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
 & & & \binom{3}{0} & & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
 \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\
 \dots & & & & & & & &
 \end{array}$$

If we replace $\binom{n}{k}$ by its value, then we obtain

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & 1 & 1 \\
 & & & & & 1 & 2 & 1 \\
 & & & 1 & 3 & 3 & 1 & \\
 & & 1 & 4 & 6 & 4 & 1 & \\
 1 & 5 & 10 & 10 & 5 & 1 & & \\
 \dots & & & & & & &
 \end{array} \tag{6}$$

Equation (6) is known as Pascal's triangle.

It turns out that the n th row of Pascal's triangle are the coefficients of the expansion of $(x+y)^n$. This is formalized by the following theorem.

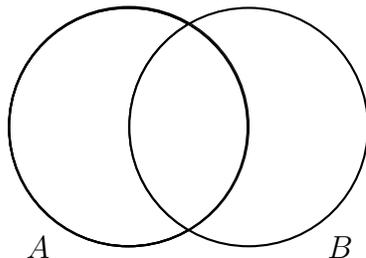
Theorem 2 (Binomial Theorem). If n is a non-negative integer, then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

We will hold off on proving this theorem until later in the course.

3.5 Inclusion-Exclusion

Many counting problems require determining the cardinality of a union $A \cup B$ of two finite sets. Although you might think that $|A \cup B| = |A| + |B|$, this is not correct. To see this, consider what happens in the case below, where A and B contain some elements in common (so $A \cap B \neq \emptyset$).



Then, $|A| + |B|$ exceeds $|A \cup B|$ by $|A \cap B|$. In other words, we have

$$|A \cup B| = |A| + |B| - |A \cap B|. \quad (7)$$

Equation (7) is sometimes called an *inclusion-exclusion* formula, because elements in $A \cap B$ are included twice in $|A| + |B|$ and then excluded when $|A \cap B|$ is subtracted. We can generalize this to multiple sets. For example, given sets A, B , and C , then you might think that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

is the correct formula. However, we have subtracted $|A \cap B \cap C|$ too many times. So, we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \quad (8)$$

Note that if $A \cap B = \emptyset$ (so $|A \cap B| = 0$), then $|A \cup B| = |A| + |B|$, and, conversely, if $|A \cup B| = |A| + |B|$, then $A \cap B = \emptyset$.

We can repeat this argument for any number of sets and thus obtain the **addition principle**.

Fact (Addition Principle). If A_1, A_2, \dots, A_n are sets with $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|.$$

Example: How many 7-digit binary strings begin in 1 or end in 1 or have exactly four ones?

Solution: Let A be the set of such strings that begins in 1, let B be the set of such strings that end in 1, and let C be the set of such strings that have exactly four ones. Then the answer to the question is $|A \cup B \cup C|$. Using Equation (8), we have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 2^6 + 2^6 + \binom{7}{4} - 2^5 - \binom{6}{3} - \binom{6}{3} + \binom{5}{2} \\ &= 64 + 64 + 35 - 32 - 20 - 20 + 10 \\ &= 101. \end{aligned}$$