Euler-Cauchy Using Undetermined Coefficients

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Introduction

- In most differential equations courses, the homogeneous second order Euler-Cauchy equation,

$$t^2y'' + aty' + by = 0, t \neq 0, \tag{1}$$

  is one of the first higher order differential equations (DEs) with variable coefficients students see.
- Some students (to my surprise) applied undetermined coefficients to directly solve certain exam problems involving nonhomogeneous Euler-Cauchy equations.
- Questions:
  1. Can we find a particular solution to this equation using substitution similar to standard undetermined coefficients?
  2. If so, when?

Review: Euler-Cauchy Equation

- The form of (1) leads us to seek solutions of the form $y(t) = t^\lambda$, where λ is a constant to be determined.
- Plugging this into (1), gives the characteristic equation: $\lambda^2 + (a-1)\lambda + b = 0$, to be solved for λ.
- Result:
  - If $(a-1)^2 > 4b$, $y(t) = c_1|t|^{\lambda_1} + c_2|t|^{\lambda_2}$.
  - If $(a-1)^2 = 4b$, $y(t) = c_1|t|^{\lambda} + c_2|t|^{\lambda}\ln|t|$.
  - If $(a-1)^2 < 4b$, let $\lambda_{1,2} = \alpha \pm i\beta$; then $y(t) = |t|^\alpha\left(c_1\cos(\beta\ln|t|) + c_2\sin(\beta\ln|t|)\right)$.

Review: Undetermined Coefficients

- Always applicable only to constant-coefficient DEs with certain right-hand side functions.
- Idea: guess the form of the particular solution y_p based on the type of right-hand side function. For example:
  - for an exponential, ae^{kt}, guess $y_p = Ae^{kt}$;
  - for a polynomial (or monomial) of degree n, guess $y_p = C_0 + C_1 t + \ldots + C_n t^n$ (a polynomial of the same degree).
- Multiply y_p by t until it contains no part of the complementary solution.
- Plug y_p into the DE and solve for the constant(s).

Euler-Cauchy and Constant-Coefficient Equations

- Assume that our Euler-Cauchy equation is given as
$$t^2 y'' + aty' + by = f(t), t > 0.$$
- Change of variables: define $t = e^z$.
- Result:
$$\frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = f(e^z),$$
a constant-coefficient DE.
- Thus, if $f(e^z)$ is one of the "special" right-hand side functions, can apply undetermined coefficients to the transformed DE.
- Leads to a method of undetermined coefficients for the original equation.

Second Order Euler-Cauchy with Monomial Right-Hand Side

- Consider the second order Euler-Cauchy equation with a monomial right-hand side function:

$$t^2y'' + aty' + by = At^\alpha, t > 0, \tag{2}$$

  where α is a real number.
- Three possibilities:
  - Case 1: α is not a root of the characteristic equation,
  - Case 2: α is a root of multiplicity one, or
  - Case 3: α is a double root.

Case 1: α is not a root of the characteristic equation

- Try as our particular solution a monomial of degree α, $y_p(t) = Ct^\alpha$.
- y_p contains no solution of the complementary equation, so keep going.
- Plug y_p into (2):

$$(\alpha(\alpha - 1) + a\alpha + b)Ct^\alpha = At^\alpha.$$

- Since α is not a root of the characteristic equation and $t \neq 0$, obtain a unique solution for C.

Case 2: α is a root of multiplicity one

- Recall: the Euler-Cauchy equation can be transformed into a constant-coefficient equation by the change of variables $t = e^z$.
- First guess for the particular solution of the transformed equation would be $y_p(z) = Ce^{\alpha z}$.
- Since α is a root of the characteristic equation, we need to multiply by z.
- Translates into multiplication by $\ln(t)$ in the particular solution for (2), so $y_p(t) = Ct^\alpha(\ln(t))$.

Case 3: α is a double root

- Similar to Case 2: look at the constant-coefficient equation.
- First guess for the particular solution of the transformed equation would be $y_p(z) = Ce^{\alpha z}$.
- Since α is a double root of the characteristic equation, we need to multiply by z^2.
- Translates into multiplication by $(\ln(t))^2$ in the particular solution for (2), so $y_p(t) = Ct^{\alpha}(\ln(t))^2$.

Summary

Theorem

For the second order Euler-Cauchy problem,

$$t^2 y'' + aty' + by = At^{\alpha}, t > 0,$$

where $\alpha \in \mathbb{R}$, a particular solution is of the form

(i) $y_p(t) = Ct^{\alpha}$, *provided that α is not equal to any root of the characteristic equation, or*

(ii) $y_p(t) = Ct^{\alpha}(\ln(t))^i$, *if α is equal to a root of the characteristic equation, where i is the multiplicity of the root.*

Example: Find a general solution of $t^2y'' - 4ty' + 4y = 4t^3 - 2t$, $t > 0$.

- Complementary solution: $y_c = c_1 t + c_2 t^4$.
- Particular solution: Solve $t^2y'' - 4ty' + 4y = 4t^3 - 2t$.
  - Use superposition to apply Theorem 1 to each part of right-hand side.
  - Guess for y_p: $y_p = At^3 + Bt\ln(t)$.
  - Plug in and collect terms: $-2At^3 - 3Bt = 4t^3 - 2t$.
  - Result: $y_p = -2t^3 + \frac{2}{3}t\ln(t)$.
- General solution: $y = y_c + y_p$, so

$$\boxed{y(t) = c_1 t + c_2 t^4 - 2t^3 + \frac{2}{3}t\ln(t)}.$$

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Right-Hand Side a Product of a Monomial and a Positive Integer Power of $\ln(t)$

- Can apply above approach to Euler-Cauchy problems with right-hand side function of the form $At^{\alpha}(\ln(t))^{n}$, $n \in \mathbb{Z}^{+}$.
- $f(e^{z})$ in the transformed equation is then $Az^{n}e^{\alpha z}$.
- Guess for the particular solution is of the form $y_p(z) = \left(C_0 + C_1 z + \ldots + C_n z^n\right) e^{\alpha z}$.
- Substitute $z = \ln(t)$ to obtain $y_p(t)$.

Result – Right-Hand Side a Product of a Monomial and a Positive Integer Power of ln(t)

Theorem

For the second order Euler-Cauchy problem,

$$t^2 y'' + aty' + by = At^{\alpha}(\ln(t))^n, t > 0,$$

where $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{Z}^+$*, a particular solution is of the form*

$$y_p(t) = (C_0 + C_1 \ln(t) + \ldots + C_n(\ln(t))^n)\, t^{\alpha}.$$

Example: Find a general solution of $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, t > 0.$

- Complementary solution: $y_c = c_1 t + c_2 t^4$.
- Particular solution: Solve $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t$.
  - Form: $y_p = y_{p_1} + y_{p_2}$, where
    $y_{p_1} = \left(A + B(\ln(t)) + C(\ln(t))^2\right) t^2$ (by Theorem 2),
    $y_{p_2} = Dt\ln(t)$ (by Theorem 1).
  - Plug y_p into the DE, collect terms, and equate coefficients to get $y_p = \left(-3 + 2\ln(t) - 2(\ln(t))^2\right) t^2 + \frac{1}{3} t\ln(t)$.
- General solution: $y = y_c + y_p$, so

$$\boxed{y(t) = c_1 t + c_2 t^4 + \left(-3 + 2\ln(t) - 2(\ln(t))^2\right) t^2 + \frac{1}{3} t\ln(t)}.$$

Example: Find a general solution of $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, t > 0.$

- Complementary solution: $y_c = c_1 t + c_2 t^4$.
- Particular solution: Solve $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t$.
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- General solution: $y = y_c + y_p$, so

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Example: Find a general solution of $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, t > 0.$

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    $y_{p_2} = Dt\ln(t)$ (by Theorem 1).
  - Plug y_p into the DE, collect terms, and equate coefficients to get $y_p = \left(-3 + 2\ln(t) - 2(\ln(t))^2\right) t^2 + \frac{1}{3} t\ln(t)$.
- General solution: $y = y_c + y_p$, so

$$y(t) = c_1 t + c_2 t^4 + \left(-3 + 2\ln(t) - 2(\ln(t))^2\right) t^2 + \frac{1}{3} t\ln(t).$$

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- Particular solution: Solve $t^2 y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t$.
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Other Right-Hand Side Functions

Easily verified that the above approach also leads to a method of undetermined coeffficients for Euler-Cauchy equations with the following right-hand side functions:

(1) $A\cos(k \ln t)$ or $A\sin(k \ln t)$,

(2) $At^{\alpha}\cos(k \ln t)$ or $At^{\alpha}\sin(k \ln t)$, and

(3) $At^{\alpha}(\ln(t))^{n}\cos(k \ln t)$ or $At^{\alpha}(\ln(t))^{n}\sin(k \ln t)$.

Conclusions

- Straightforward to generalize this approach to higher order Euler-Cauchy equations.
- This “new" approach makes a good addition to the discussion of Euler-Cauchy problems in a differential equations course.