Modeling with First-Order ODEs

CURM Background Material, Fall 2014

Equilibrium Points and Stability -- the Logistic Model

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$r := 2; K := 10;$

$$2$$
$$10 \quad \textbf{(1.1)}$$

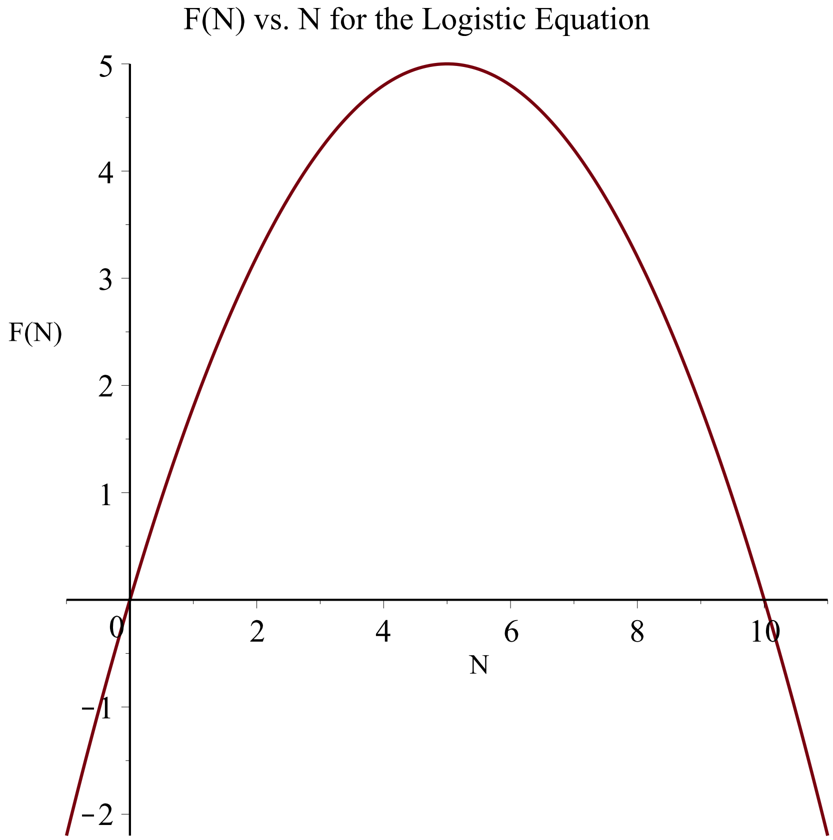
First, we determine the equilibria by letting $N' = 0$.

$solve\left(r \cdot N \cdot \left(1 - \frac{N}{K}\right) = 0\right)$

$$0, 10 \quad \textbf{(1.2)}$$

Now, let's analyze the stability of the equilibria. First, plot the right-hand side of the DE.

$plot\left(r \cdot N \cdot \left(1 - \frac{N}{K}\right), N = -1 .. K + 1, title = \text{"F(N) vs. N for the Logistic Equation"}, labels = [\text{"N"}, \text{"F(N)"}]\right)$

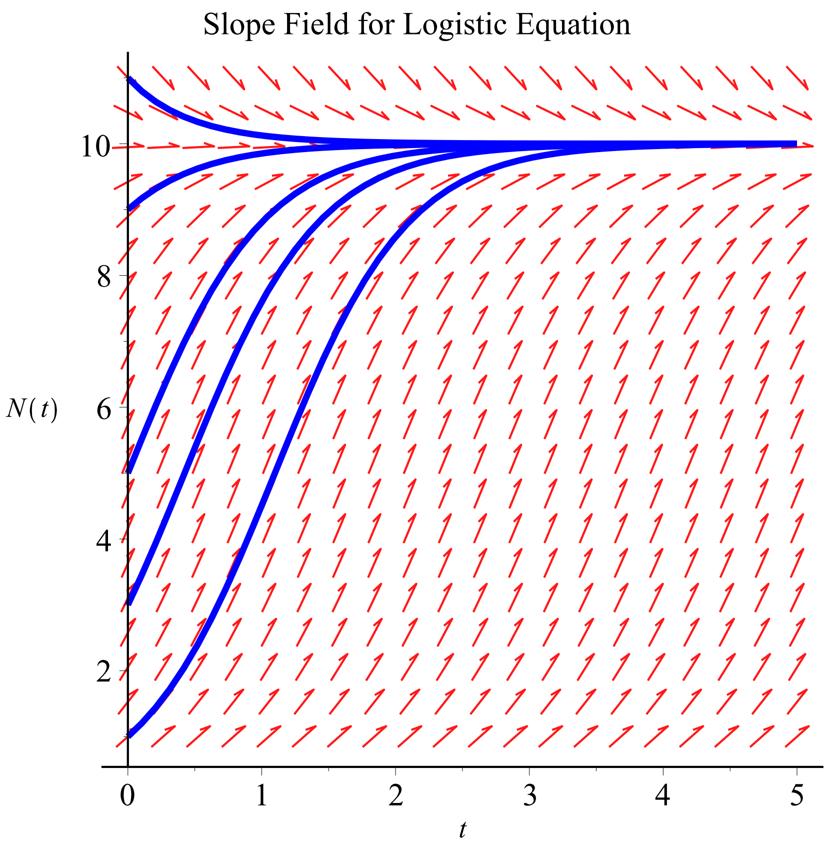


Notice that $F(N)$ is positive for all $0 < N < K$, so N is increasing, and $F(N) < 0$ for $N > K$, so N is decreasing. Thus, $N = K$ is stable.

Let's look at the slope field to qualitatively analyze the solution. I use an unusually small time period so that you can see the details in the slope field.

$with(DEtools):$

$DEplot\left(diff(N(t), t) = r \cdot N(t) \cdot \left(1 - \frac{N(t)}{K}\right), N(t), t = 0..5, [[N(0) = 1], [N(0) = 3], [N(0) = 5], [N(0) = 9], [N(0) = 11]], linecolor = blue, title = \text{"Slope Field for Logistic Equation"} \right)$



Notice that all solution curves tend towards $K = 10$, again showing the $N = K$ is stable. All nonnegative solutions tend away from 0, showing that $N = 0$ is unstable.

Solving Analytically -- the Logistic Model

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$$Pop := t \rightarrow dsolve\left(diff(N(t), t) = r \cdot N(t) \cdot \left(1 - \frac{N(t)}{K}\right)\right)$$

$$t \rightarrow dsolve\left(\frac{\mathrm{d}}{\mathrm{d}t} N(t) = r\, N(t) \left(1 - \frac{N(t)}{K}\right)\right) \qquad \textbf{(2.1)}$$

$$assume(r > 0);\ \lim_{t \rightarrow \infty} Pop(t)$$

$$\lim_{t \rightarrow \infty} N(t) = K \qquad \textbf{(2.2)}$$