Example: Analysis of Equilibrium Solutions for a Nonlinear System

Hardwood and Softwood Trees

CURM Background Material, Fall 2014

Step 6: Solve and Interpret the Model

Equilibrium Solutions

restart

$f1 := (H, S) \to 0.1 \cdot H \cdot \left(1 - \frac{H}{10000}\right) - b_1 \cdot S \cdot H;$

$f2 := (H, S) \to 0.25 \cdot S \cdot \left(1 - \frac{S}{6000}\right) - b_2 \cdot S \cdot H;$

$$(H, S) \to 0.1\, H \left(1 - \frac{1}{10000}\, H\right) - b_1\, S\, H$$

$$(H, S) \to 0.25\, S \left(1 - \frac{1}{6000}\, S\right) - b_2\, S\, H \quad \textbf{(1.1.1)}$$

$sol := solve(\{f1(H, S) = 0, f2(H, S) = 0\}, \{H, S\})$

$$\{H = 0., S = 0.\}, \{H = 0., S = 6000.\}, \{H = 10000., S = 0.\}, \left\{H = \frac{10000.\ (-1. + 60000.\ b_1)}{2.400000000\ 10^9\, b_1\, b_2 - 1.}, S = \frac{6000.\ (40000.\ b_2 - 1.)}{2.400000000\ 10^9\, b_1\, b_2 - 1.}\right\} \quad \textbf{(1.1.2)}$$

There is only one equilibrium solution for which both populations are nonzero.

$Heb := rhs(sol[4][1]);\ Seb := rhs(sol[4][2]);$

$$\frac{10000.\ (-1. + 60000.\ b_1)}{2.400000000\ 10^9\, b_1\, b_2 - 1.}$$

$$\frac{6000.\ (40000.\ b_2 - 1.)}{2.400000000\ 10^9\, b_1\, b_2 - 1.} \quad \textbf{(1.1.3)}$$

$solve\left(10000-6\cdot 10^{8}\cdot b_1 > 0\right)$

$$RealRange\left(-\infty, Open\left(\frac{1}{60000}\right)\right) \quad \textbf{(1.1.4)}$$

$solve\left(6000 - 2.4\cdot 10^{8}\cdot b_2 > 0\right)$

$$RealRange(-\infty, Open(0.00002500000000)) \quad \textbf{(1.1.5)}$$

$0.000025\cdot 6000$

$$0.150000 \quad \textbf{(1.1.6)}$$

Stability Analysis Using Linearization

Determine the entries into the Jacobian matrix (necessary for stability analysis).

$df1H := unapply(diff(f1(H, S), H), H, S);$
$df1S := unapply(diff(f1(H, S), S), H, S);$

$$(H, S) \rightarrow 0.1 - 0.00002000000000\, H - b_1\, S$$

$$(H, S) \rightarrow -b_1\, H \quad \textbf{(1.2.1)}$$

$df2H := unapply(diff(f2(H, S), H), H, S);$
$df2S := unapply(diff(f2(H, S), S), H, S);$

$$(H, S) \rightarrow -b_2\, S$$

$$(H, S) \rightarrow 0.25 - 0.00008333333333\, S - b_2\, H \quad \textbf{(1.2.2)}$$

Evaluate the Jacobian of the functions *f1* and *f2* at the nonzero equilibrium solution and determine its eigenvalues.

$$A := \begin{bmatrix} df1H(Heb, Seb) & df1S(Heb, Seb) \\ df2H(Heb, Seb) & df2S(Heb, Seb) \end{bmatrix}$$

$$\left[\left[0.1 - \frac{0.2000000000\,(-1. + 60000.\, b_1)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.} - \frac{6000.\, b_1\,(40000.\, b_2 - 1.)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.}, -\frac{10000.\, b_1\,(-1. + 60000.\, b_1)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.}\right], \left[-\frac{6000.\, b_2\,(40000.\, b_2 - 1.)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.}, 0.25 - \frac{0.5000000000\,(40000.\, b_2 - 1.)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.} - \frac{10000.\, b_2\,(-1. + 60000.\, b_1)}{2.400000000\, 10^{9}\, b_1\, b_2 - 1.}\right]\right] \quad \textbf{(1.2.3)}$$

$simplify(\%)$

$$\begin{bmatrix} \frac{0.1000000000 - 6000.\, b_1}{2.400000000\, 10^9\, b_1\, b_2 - 1.} & -\frac{10000.\, b_1\, (-1. + 60000.\, b_1)}{2.400000000\, 10^9\, b_1\, b_2 - 1.} \\ -\frac{6000.\, b_2\, (40000.\, b_2 - 1.)}{2.400000000\, 10^9\, b_1\, b_2 - 1.} & \frac{0.2500000000 - 10000.\, b_2}{2.400000000\, 10^9\, b_1\, b_2 - 1.} \end{bmatrix} \quad \textbf{(1.2.4)}$$

with(*LinearAlgebra*)

[*&x*, *Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*, *BidiagonalForm*, *BilinearForm*, *CARE*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *Column*, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*, *CompanionMatrix*, *CompressedSparseForm*, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*, *Copy*, *CreatePermutation*, *CrossProduct*, *DARE*, *DeleteColumn*, *DeleteRow*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *DotProduct*, *EigenConditionNumbers*, *Eigenvalues*, *Eigenvectors*, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *FromCompressedSparseForm*, *FromSplitForm*, *GaussianElimination*, *GenerateEquations*, *GenerateMatrix*, *Generic*, *GetResultDataType*, *GetResultShape*, *GivensRotationMatrix*, *GramSchmidt*, *HankelMatrix*, *HermiteForm*, *HermitianTranspose*, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*, *IdentityMatrix*, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *KroneckerProduct*, *LA_Main*, *LUDecomposition*, *LeastSquares*, *LinearSolve*, *LyapunovSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *ProjectionMatrix*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SplitForm*, *StronglyConnectedBlocks*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *SylvesterSolve*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*] **(1.2.5)**

$p := \lambda \rightarrow CharacteristicPolynomial(A, \lambda)$

$$\lambda \rightarrow LinearAlgebra{:-}CharacteristicPolynomial(A, \lambda) \quad \textbf{(1.2.6)}$$

$eigs := solve(p(\lambda) = 0, \lambda)$

$$\frac{1}{2.400000000\, 10^9\, b_1\, b_2 - 1.} \Big(0.02500000000 \Big(-1.20000\, 10^5\, b_1 - 2.00000\, 10^5\, b_2 + 7. \quad \textbf{(1.2.7)}$$

$$+ \Big(2.304000000\, 10^{20}\, b_1^2\, b_2^2 - 5.760000000\, 10^{15}\, b_1^2\, b_2$$

$$- 3.840000000\, 10^{15}\, b_1\, b_2^2 + 1.440000000\, 10^{10}\, b_1^2 + 4.800000000\, 10^{10}\, b_1\, b_2$$

$+ 4.000000000\ 10^{10}\ b_2^2 + 7.20000\ 10^5\ b_1 - 1.200000\ 10^6\ b_2 + 9.)^{1/2}\Big)\Big),$

$- \frac{1}{2.400000000\ 10^9\ b_1\ b_2 - 1.} \Big(0.02500000000\ \Big(1.20000\ 10^5\ b_1 + 2.00000\ 10^5\ b_2$

$- 7.$

$+ \big(2.304000000\ 10^{20}\ b_1^2\ b_2^2 - 5.760000000\ 10^{15}\ b_1^2\ b_2$

$- 3.840000000\ 10^{15}\ b_1\ b_2^2 + 1.440000000\ 10^{10}\ b_1^2 + 4.800000000\ 10^{10}\ b_1\ b_2$

$+ 4.000000000\ 10^{10}\ b_2^2 + 7.20000\ 10^5\ b_1 - 1.200000\ 10^6\ b_2 + 9.)^{1/2}\Big)\Big)$

$r1 := eigs[1]$

$\frac{1}{2.400000000\ 10^9\ b_1\ b_2 - 1.} \Big(0.02500000000\ \Big(-1.20000\ 10^5\ b_1 - 2.00000\ 10^5\ b_2 + 7.$ **(1.2.8)**

$+ \big(2.304000000\ 10^{20}\ b_1^2\ b_2^2 - 5.760000000\ 10^{15}\ b_1^2\ b_2$

$- 3.840000000\ 10^{15}\ b_1\ b_2^2 + 1.440000000\ 10^{10}\ b_1^2 + 4.800000000\ 10^{10}\ b_1\ b_2$

$+ 4.000000000\ 10^{10}\ b_2^2 + 7.20000\ 10^5\ b_1 - 1.200000\ 10^6\ b_2 + 9.)^{1/2}\Big)\Big)$

$r2 := eigs[2]$

$- \frac{1}{2.400000000\ 10^9\ b_1\ b_2 - 1.} \Big(0.02500000000\ \Big(1.20000\ 10^5\ b_1 + 2.00000\ 10^5\ b_2 - 7.$ **(1.2.9)**

$+ \big(2.304000000\ 10^{20}\ b_1^2\ b_2^2 - 5.760000000\ 10^{15}\ b_1^2\ b_2$

$- 3.840000000\ 10^{15}\ b_1\ b_2^2 + 1.440000000\ 10^{10}\ b_1^2 + 4.800000000\ 10^{10}\ b_1\ b_2$

$+ 4.000000000\ 10^{10}\ b_2^2 + 7.20000\ 10^5\ b_1 - 1.200000\ 10^6\ b_2 + 9.)^{1/2}\Big)\Big)$

$assume\left(0 < b1, b1 < \frac{1}{60000}, 0 < b2, b2 < \frac{1}{40000}\right)$

$is\big(1.440000000\ 10^{10}\ b_1^2 + 7.20000\ 10^5\ b_1 + 4.800000000\ 10^{10}\ b_2\ b_1 + 9.$
$- 1.200000\ 10^6\ b_2 + 4.000000000\ 10^{10}\ b_2^2 - 3.840000000\ 10^{15}\ b_2^2\ b_1$
$+ 2.304000000\ 10^{20}\ b_2^2\ b_1^2 - 5.760000000\ 10^{15}\ b_2\ b_1^2, positive\big)$

false **(1.2.10)**

$coulditbe\big(1.440000000\ 10^{10}\ b_1^2 + 7.20000\ 10^5\ b_1 + 4.800000000\ 10^{10}\ b_2\ b_1 + 9.$
$- 1.200000\ 10^6\ b_2 + 4.000000000\ 10^{10}\ b_2^2 - 3.840000000\ 10^{15}\ b_2^2\ b_1$
$+ 2.304000000\ 10^{20}\ b_2^2\ b_1^2 - 5.760000000\ 10^{15}\ b_2\ b_1^2, positive\big)$

true **(1.2.11)**

$coulditbe\big(1.440000000\ 10^{10}\ b_1^2 + 7.20000\ 10^5\ b_1 + 4.800000000\ 10^{10}\ b_2\ b_1 + 9.$

$$- 1.200000\ 10^{6}\ b_2 + 4.000000000\ 10^{10}\ b_2^2 - 3.840000000\ 10^{15}\ b_2^2\ b_1 + 2.304000000\ 10^{20}\ b_2^2\ b_1^2 - 5.760000000\ 10^{15}\ b_2\ b_1^2, negative)$$

$$true \qquad (1.2.12)$$

$$coulditbe(1.440000000\ 10^{10}\ b_1^2 + 7.20000\ 10^{5}\ b_1 + 4.800000000\ 10^{10}\ b_2\ b_1 + 9. - 1.200000\ 10^{6}\ b_2 + 4.000000000\ 10^{10}\ b_2^2 - 3.840000000\ 10^{15}\ b_2^2\ b_1 + 2.304000000\ 10^{20}\ b_2^2\ b_1^2 - 5.760000000\ 10^{15}\ b_2\ b_1^2, 0)$$

$$true \qquad (1.2.13)$$

$$is(-1.20000\ 10^{5}\ b_1 + 7. - 2.00000\ 10^{5}\ b_2, positive)$$

$$false \qquad (1.2.14)$$

$$coulditbe(-1.20000\ 10^{5}\ b_1 + 7. - 2.00000\ 10^{5}\ b_2, positive)$$

$$true \qquad (1.2.15)$$

$$coulditbe(-1.20000\ 10^{5}\ b_1 + 7. - 2.00000\ 10^{5}\ b_2, negative)$$

$$true \qquad (1.2.16)$$

$$coulditbe(-1.20000\ 10^{5}\ b_1 + 7. - 2.00000\ 10^{5}\ b_2, 0)$$

$$true \qquad (1.2.17)$$

$$solve(-1.20000\ 10^{5}\ b_1 + 7. - 2.00000\ 10^{5}\ b_2 > 0, \{b_1, b_2\})$$

$$\{b_2 = b_2, b_1 < 0.00005833333333 - 1.666666667\ b_2\} \qquad (1.2.18)$$

Step 7: Validate the Model

restart

$$b_1 := \frac{t}{60000}; b_2 := \frac{t}{40000};$$

$$\frac{1}{60000}\ t$$

$$\frac{1}{40000}\ t \qquad (2.1)$$

$$solve\left(\left\{b_1 < \frac{1}{60000}, b_2 < \frac{1}{40000}, b_2 < -0.6 \cdot b_1 + 0.000035\right\}, t\right)$$

$$\{t < 1.\} \qquad (2.2)$$

$$f1 := (H, S, t) \rightarrow 0.10 \cdot H \cdot \left(1 - \frac{H}{10000}\right) - b_1 \cdot S \cdot H;$$

$$f2 := (H, S, t) \rightarrow 0.25 \cdot S \cdot \left(1 - \frac{S}{6000}\right) - b_2 \cdot S \cdot H;$$

$$(H, S, t) \rightarrow 0.10\ H \left(1 - \frac{1}{10000}\ H\right) - b_1\ S\ H$$

$$(H, S, t) \to 0.25\, S \left(1 - \frac{1}{6000}\, S\right) - b_2\, S\, H \qquad \textbf{(2.3)}$$

Equilibrium Solutions

sol := *solve*({ *f1*(*H*, *S*, *t*) = 0, *f2*(*H*, *S*, *t*) = 0}, { *H*, *S*})

$$\{H = 0., S = 0.\}, \{H = 0., S = 6000.\}, \{H = 10000., S = 0.\}, \left\{H = \frac{10000.}{t + 1.}, S = \frac{6000.}{t + 1.}\right\} \qquad \textbf{(2.1.1)}$$

Heb := *rhs*(*sol*[4][1]); *Seb* := *rhs*(*sol*[4][2]);

$$\frac{10000.}{t + 1.}$$

$$\frac{6000.}{t + 1.} \qquad \textbf{(2.1.2)}$$

Since we already require $0 < t < 1$, the equilibrium solution is positive.

Stability Analysis Using Linearization

Determine the entries into the Jacobian matrix (necessary for stability analysis).

df1H := *unapply*(*diff* (*f1*(*H*, *S*, *t*), *H*), *H*, *S*);
df1S := *unapply*(*diff* (*f1*(*H*, *S*, *t*), *S*), *H*, *S*);

$$(H, S) \to 0.10 - 0.00002000000000\, H - \frac{1}{60000}\, t\, S$$

$$(H, S) \to -\frac{1}{60000}\, t\, H \qquad \textbf{(2.2.1)}$$

df2H := *unapply*(*diff* (*f2*(*H*, *S*, *t*), *H*), *H*, *S*);
df2S := *unapply*(*diff* (*f2*(*H*, *S*, *t*), *S*), *H*, *S*);

$$(H, S) \to -\frac{1}{40000}\, t\, S$$

$$(H, S) \to 0.25 - 0.00008333333333\, S - \frac{1}{40000}\, t\, H \qquad \textbf{(2.2.2)}$$

Evaluate the Jacobian of the functions *f1* and *f2* at the nonzero equilibrium solution and determine its eigenvalues.

$$A := \begin{bmatrix} df1H(Heb, Seb, t) & df1S(Heb, Seb, t) \\ df2H(Heb, Seb, t) & df2S(Heb, Seb, t) \end{bmatrix}$$

$$\left[\left[0.10 - \frac{0.2000000000}{t + 1.} - \frac{0.1000000000\, t}{t + 1.}, -\frac{0.1666666667\, t}{t + 1.}\right], \left[-\frac{0.1500000000\, t}{t + 1.}, 0.25 - \frac{0.5000000000}{t + 1.} - \frac{0.2500000000\, t}{t + 1.}\right]\right] \qquad \textbf{(2.2.3)}$$

simplify(%)

$$\begin{bmatrix} -\frac{0.1000000000}{t+1.} & -\frac{0.1666666667\, t}{t+1.} \\ -\frac{0.1500000000\, t}{t+1.} & -\frac{0.2500000000}{t+1.} \end{bmatrix} \quad (2.2.4)$$

with(*LinearAlgebra*) :
assume(0 < *t*, *t* < 1)
$p := \lambda \rightarrow CharacteristicPolynomial(A, \lambda)$

$$\lambda \rightarrow LinearAlgebra{:-}CharacteristicPolynomial(A, \lambda) \quad (2.2.5)$$

$eigs := solve(p(\lambda) = 0, \lambda)$

$$\frac{0.02500000000\left(-7. + \sqrt{40.\, t\sim^2 + 9.}\right)}{t\sim + 1.}, -\frac{0.02500000000\left(7. + \sqrt{40.\, t\sim^2 + 9.}\right)}{t\sim + 1.} \quad (2.2.6)$$

$r1 := eigs[1]$

$$\frac{0.02500000000\left(-7. + \sqrt{40.\, t\sim^2 + 9.}\right)}{t\sim + 1.} \quad (2.2.7)$$

$r2 := eigs[2]$

$$-\frac{0.02500000000\left(7. + \sqrt{40.\, t\sim^2 + 9.}\right)}{t\sim + 1.} \quad (2.2.8)$$

$is(\mathrm{r1} < 0)$

$$true \quad (2.2.9)$$

$is(\mathrm{r2} < 0)$

$$true \quad (2.2.10)$$

We can also plot the eigenvalues over the valid region for the parameter *t* to see that both eigenvalues are negative.

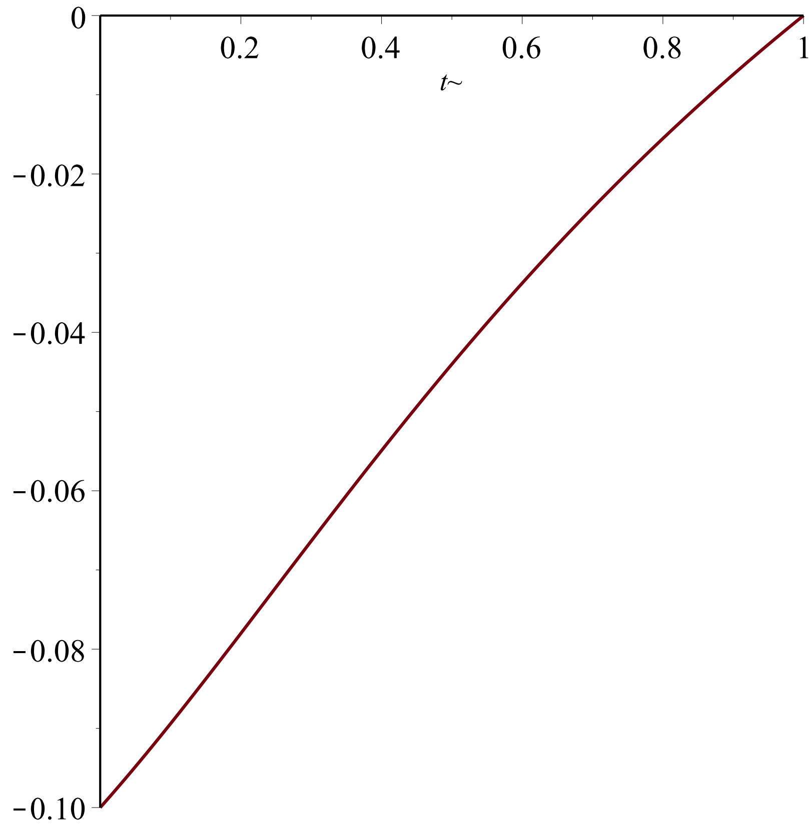
$\lambda 1 := t \rightarrow r1$

$$t \rightarrow r1 \quad (2.2.11)$$

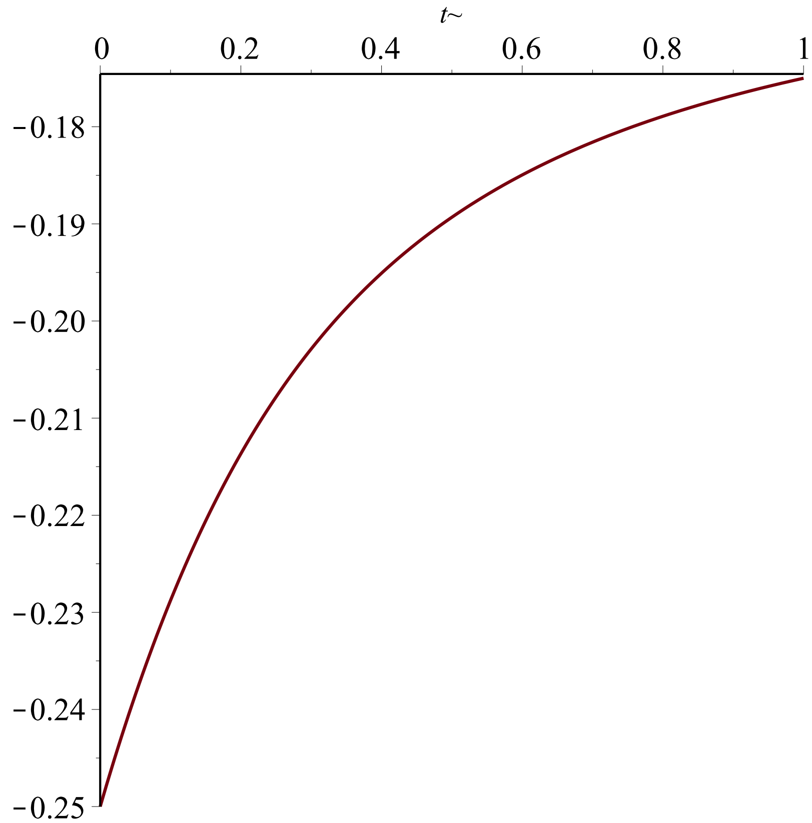
$\lambda 2 := t \rightarrow r2$

$$t \rightarrow r2 \quad (2.2.12)$$

$plot(\lambda 1(t), t = 0 ..1)$



$plot(\lambda 2(t),\ t = 0\ ..1)$


t~
0
0.2
0.4
0.6
0.8
1
−0.18
−0.19
−0.20
−0.21
−0.22
−0.23
−0.24
−0.25