Examples: Modeling Using Linear Systems of Equations

CURM Background Material, Fall 2014

Models Involving Systems with a Unique Solution

Example 1: Electrical Networks

We will solve the system of equations in two ways, first using *solve* and second, by writing the system in augmented matrix form and reducing it to reduced row echelon form.

Direct Solution Using Maple's solve Function

$solve(\{8 \cdot I1 - 3 \cdot I3 = 10, 6 \cdot I2 + 3 \cdot I3 = 15, I1 - I2 + I3 = 0\}, \{I1, I2, I3\})$

$$\left\{I1 = \frac{3}{2}, I2 = \frac{13}{6}, I3 = \frac{2}{3}\right\} \quad \textbf{(1.1.1.1)}$$

Matrix Solution

$$A := \begin{bmatrix} 8 & 0 & -3 & 10 \\ 0 & 6 & 3 & 15 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 0 & -3 & 10 \\ 0 & 6 & 3 & 15 \\ 1 & -1 & 1 & 0 \end{bmatrix} \quad \textbf{(1.1.2.1)}$$

$with(LinearAlgebra):$
$AA := ReducedRowEchelonForm(A);$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{13}{6} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix} \quad \textbf{(1.1.2.2)}$$

As you can see, the solution obtained is $I1 = \frac{3}{2}, I2 = \frac{13}{6}, I3 = \frac{2}{3}$, the same answer as above.

Example 2: Fitting a Power Curve -- Height vs. Weight (see also *regression.mw*)

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$E := (a, b, c, d) \rightarrow \left(y - \left(a \cdot x^3 + \mathrm{b} \cdot x^2 + c \cdot x + d\right)\right)^2$

$$(a, b, c, d) \rightarrow \left(y - a\, x^3 - b\, x^2 - c\, x - d\right)^2 \qquad \textbf{(1.2.1)}$$

$diff(E(a, b, c, d), a)$

$$-2\left(-a\, x^3 - b\, x^2 - c\, x - d + y\right) x^3 \qquad \textbf{(1.2.2)}$$

$diff(E(a, b, c, d), b)$

$$-2\left(-a\, x^3 - b\, x^2 - c\, x - d + y\right) x^2 \qquad \textbf{(1.2.3)}$$

$diff(E(a, b, c, d), c)$

$$-2\left(-a\, x^3 - b\, x^2 - c\, x - d + y\right) x \qquad \textbf{(1.2.4)}$$

$diff(E(a, b, c, d), d)$

$$2\, a\, x^3 + 2\, b\, x^2 + 2\, c\, x + 2\, d - 2\, y \qquad \textbf{(1.2.5)}$$

$ht := [62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]$

$$[62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74] \qquad \textbf{(1.2.6)}$$

$wt := [128, 131, 135, 139, 142, 146, 150, 154, 158, 162, 167, 172, 177]$

$$[128, 131, 135, 139, 142, 146, 150, 154, 158, 162, 167, 172, 177] \qquad \textbf{(1.2.7)}$$

$x := \sum_{i=1}^{13} ht[i]$

$$884 \qquad \textbf{(1.2.8)}$$

$x2 := \sum_{i=1}^{13} (ht[i])^2$

$$60294 \qquad \textbf{(1.2.9)}$$

$x3 := \sum_{i=1}^{13} (ht[i])^3$

$$4124744 \qquad \textbf{(1.2.10)}$$

$x4 := \sum_{i=1}^{13} (ht[i])^4$

$$283011846 \qquad \textbf{(1.2.11)}$$

$x5 := \sum_{i=1}^{13} (ht[i])^5$

$$19474949624 \qquad \textbf{(1.2.12)}$$

$$x6 := \sum_{i=1}^{13} (ht[i])^6$$

$$1343964152934 \quad \textbf{(1.2.13)}$$

$$y := \sum_{i=1}^{13} wt[i]$$

$$1961 \quad \textbf{(1.2.14)}$$

$$yx := \sum_{i=1}^{13} wt[i] \cdot ht[i]$$

$$134084 \quad \textbf{(1.2.15)}$$

$$yx2 := \sum_{i=1}^{13} wt[i] \cdot (ht[i])^2$$

$$9195356 \quad \textbf{(1.2.16)}$$

$$yx3 := \sum_{i=1}^{13} wt[i] \cdot (ht[i])^3$$

$$632459042 \quad \textbf{(1.2.17)}$$

Solution Using Maple's solve Function

$solve(\{a \cdot x6 + b \cdot x5 + c \cdot x4 + d \cdot x3 = yx3, a \cdot x5 + b \cdot x4 + c \cdot x3 + d \cdot x2 = yx2, a \cdot x4 + b \cdot x3 + c \cdot x2 + d \cdot x = yx, a \cdot x3 + b \cdot x2 + c \cdot x + d \cdot 13 = y\}, \{a, b, c, d\})$

$$\left\{a = \frac{19}{3432}, b = -\frac{163}{154}, c = \frac{1707059}{24024}, d = -\frac{3060027}{2002}\right\} \quad \textbf{(1.2.1.1)}$$

$with(plots):$

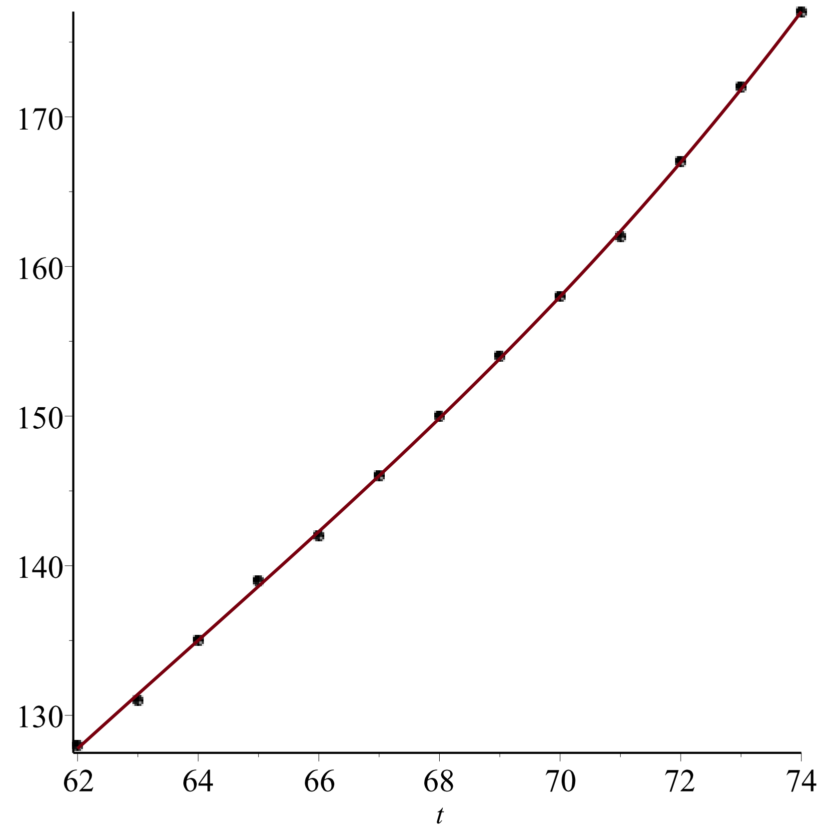
$Data := pointplot(\{seq([ht[i], wt[i]], i = 1..13)\}, symbol = solidcircle):$

$Poly := t \rightarrow \frac{19}{3432} \cdot t^3 - \frac{163}{154} \cdot t^2 + \frac{1707059}{24024} \cdot t - \frac{3060027}{2002}$

$$t \rightarrow \frac{19}{3432} t^3 - \frac{163}{154} t^2 + \frac{1707059}{24024} t - \frac{3060027}{2002} \quad \textbf{(1.2.1.2)}$$

$Polyfit := plot(Poly(t), t = 62..74):$

$display(\{Data, Polyfit\})$



Matrix Solution

$$A := \begin{bmatrix} x6 & x5 & x4 & x3 & yx3 \\ x5 & x4 & x3 & x2 & yx2 \\ x4 & x3 & x2 & x & yx \\ x3 & x2 & x & 13 & y \end{bmatrix}$$

$$\begin{bmatrix} 1343964152934 & 19474949624 & 283011846 & 4124744 & 632459042 \\ 19474949624 & 283011846 & 4124744 & 60294 & 9195356 \\ 283011846 & 4124744 & 60294 & 884 & 134084 \\ 4124744 & 60294 & 884 & 13 & 1961 \end{bmatrix} \quad \textbf{(1.2.2.1)}$$

with(*LinearAlgebra*) :
ReducedRowEchelonForm(*A*)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{19}{3432} \\ 0 & 1 & 0 & 0 & -\frac{163}{154} \\ 0 & 0 & 1 & 0 & \frac{1707059}{24024} \\ 0 & 0 & 0 & 1 & -\frac{3060027}{2002} \end{bmatrix} \quad (1.2.2.2)$$

As you can see, the values obtained for the variables are the same:
$a = \frac{19}{3432}$, $b = -\frac{163}{154}$, $c = \frac{1707059}{24024}$, $d = -\frac{3060027}{2002}$.

Models Involving Systems with Infinite Solutions

Example 1: Leontief's "Exchange Model"

Solution Using Maple's solve Function

$solve(\{pC = 0.1 \cdot pC + 0.4 \cdot pE + 0.6 \cdot pS, pE = 0.5 \cdot pC + 0.1 \cdot pE + 0.2 \cdot pS, pS = 0.4 \cdot pC + 0.5 \cdot pE + 0.2 \cdot pS\}, \{pC, pE, pS\})$

$$\{pC = 1.016393443\, pS, pE = 0.7868852459\, pS, pS = pS\} \quad (2.1.1.1)$$

Matrix Solution

Note that the above is a homogeneous system of equations, so we may solve by writing the coefficient matrix.

$$A := \begin{bmatrix} 0.9 & -0.4 & -0.6 \\ -0.5 & 0.9 & -0.2 \\ -0.4 & -0.5 & 0.8 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & -0.4 & -0.6 \\ -0.5 & 0.9 & -0.2 \\ -0.4 & -0.5 & 0.8 \end{bmatrix} \quad (2.1.2.1)$$

$with(LinearAlgebra):$
$ReducedRowEchelonForm(A)$

$$\begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix} \quad (2.1.2.2)$$

This is incorrect, since the determinant of A is 0. Therefore, you want to be careful using this function in Maple.

$Determinant(A)$

$$0. \qquad (2.1.2.3)$$

Let's see what happens if we use the *GaussianElimination* function.

$A1 := GaussianElimination(A)$

$$\begin{bmatrix} 0.900000000000000 & -0.400000000000000 & -0.600000000000000 \\ 0. & 0.677777777777778 & -0.533333333333333 \\ 0. & 0. & 1.11022302462516\ 10^{-16} \end{bmatrix} \qquad (2.1.2.4)$$

You can see that the entry in the third row and third column is essentially 0. It is not exactly 0 due to roundoff error.

Example 2: Balancing Chemical Equations

Solution Using Maple's solve Function

$restart$
$solve(\{x1 - 6 \cdot x4 = 0, 2 \cdot x1 + x2 - 2 \cdot x3 - 6 \cdot x4 = 0, 2 \cdot x2 - 12 \cdot x4 = 0\}, \{x1, x2, x3, x4\})$

$$\{x1 = 6\, x4, x2 = 6\, x4, x3 = 6\, x4, x4 = x4\} \qquad (2.2.1.1)$$

Matrix Solution

$$A := \begin{bmatrix} 1 & 0 & 0 & -6 \\ 2 & 1 & -2 & -6 \\ 0 & 2 & 0 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 2 & 1 & -2 & -6 \\ 0 & 2 & 0 & -12 \end{bmatrix} \qquad (2.2.2.1)$$

$with(LinearAlgebra):$
$ReducedRowEchelonForm(A)$

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -6 \end{bmatrix} \qquad (2.2.2.2)$$