Modeling First-Order Ordinary Differential Equations (ODEs)

CURM Background Material, Fall 2014
Dr. Doreen De Leon

1 Examples of First-Order Models

1.1 Radioactive Decay

In Calculus II, you probably saw the model for radioactive decay. We will look at this model in a bit more detail now. First, we make the following assumptions:

1. We have an object that will emit radiation if exposed to a certain substance.
2. The emissions are all of the same kind and they all decay by the same process (alpha, beta, gamma emissions).
3. The decaying process is such that once a particular nucleus decays, it cannot repeat the process again.
4. The probability of a given nucleus decaying in any time interval is constant.

If we have N undecayed nuclei present at time t, the number of nuclei that will decay between t and $t + \triangle t$ must be proportional to the product of N and $\triangle t$; i.e.,

$$\triangle N = -kN\triangle t,$$

where $k > 0$ is the constant of proportionality, known as the decay constant. If we assume that the number of nuclei N is a continuous function of time, we can divide both sides of the equation by $\triangle t$ and take the limit as $\triangle t \to 0$ to obtain the standard radioactive decay model

$$\frac{dN}{dt} = -kN.$$

The decay constant k is determined by experimental data. This equation also holds if we consider the mass of radioactive element, instead of the number of nuclei (the total mass is just the product of the mass of one atom multiplied by the number of atoms, and such constant may be eliminated from the equation).

This is a separable differential equation, and its solution is

$$N(t) = N_0 e^{-kt},$$

where N_0 is the initial amount of radioactive element.

1.2 Newton's Law of Cooling

Newton's law of cooling states that the surface temperature of an object changes at a rate proportional to the difference between the ambient temperature and the temperature of the object. So, if $T(t)$ represents the temperature of an object at time t and T_a represents the ambient temperature, we have

$$\frac{dT}{dt} = k(T_a - T).$$

This is also a separable differential equation, with solution

$$T(t) = T_a + ce^{-kt},$$

where c is determined by the initial temperature of the object.

1.3 Population Models

Consider the population of one species in a specified region, e.g., the number of pepole in the world, the number of pine trees in a forest, the number of bacteria in an experiment. We will ignore differences in the individuals comprising the group (e.g., male-female, age). In formulating a model of the population growth of a species, we must decide what factors affect that population. In some cases, the population size depends on many quantities.

Example: Sharks in the Adriatic Sea. What factors can affect the population size of sharks in the Adriatic Sea?

1. The number of fish available for the sharks to consume.
2. The presence of a harmful bacteria.
3. Water temperature and salinity may be important factors, as well.

1.3.1 Exponential Growth

Suppose measurements of the population size of a species (not affected by any other) are taken over an interval of time $\triangle t$. The rate of change of the population measured over the time interval $\triangle t$ is

$$\frac{\triangle N}{\triangle t} = \frac{N(t + \triangle t) - N(t)}{\triangle t}.$$

The growth rate per unit of time $R(t)$ is the rate of change of the population per individual, i.e.,

$$R(t) = \frac{N(t + \triangle t) - N(t)}{\triangle t N(t)}.$$

Since the percent change in the population is $100\frac{\triangle N(t)}{N(t)} = 100R(t)\triangle t$, $R(t)$ is the percentage change in the population per unit time.

In general, the growth rate can depend on time and is calculated over a time interval of length $\triangle t$. By its definition, the growth rate depnds on the measuring time interval. We will now approximate the population as a continuous, differentiable function of time. So, we can determine the instantaneous growth rate:

$$\begin{aligned}
R(t) &= \lim_{\triangle t \to 0} \frac{N(t+\triangle t) - N(t)}{\triangle t N(t)} \\
&= \frac{1}{N(t)} \lim_{\triangle t \to 0} \frac{N(t+\triangle t) - N(t)}{\triangle t} \\
&= \frac{1}{N(t)} \frac{dN(t)}{dt}.
\end{aligned}$$

Note: Approximating the population as a continuous, differentiable function is most reasonable for large populations.

The first model of population size, then, is to assume that the growth rate is constant, say $R(t) = R_0$. Then we obtain

$$\frac{dN}{dt} = R_0 N,$$

which is similar to the radioactive decay model. If we are given a population at t_0 (not necessarily 0), $N(t_0) = N_0$, then the solution to this equation is

$$N(t) = N_0 e^{R_0(t-t_0)}.$$

1.3.2 More Realism – Self-limiting Growth

The assumption of constant population growth is very unrealistic. It ignores the impact of environmental factors such as climate, food suply, and economic factors. In general, the growh rate $\frac{1}{N}\frac{dN}{dt}$ may not be constant but might depend upon the population size:

$$\frac{1}{N}\frac{dN}{dt} = R(N), \text{ or } \frac{dN}{dt} = NR(N).$$

What are the properties of $R(N)$?

- For moderate-sized populations, growth occurs with only slight limitations from the species' environment.
- As N diminishes, $R(N)$ should approach the growth rate without environmental influences.
- As the population size increases, it is expected to grow at a smaller rate due to limitations on growth caused by an increase in population density, so $R(N)$ decreases as N increases.
- For simplicity, we will model growth rate for very small populations in the same way.

The simplest function satisfying the above properties is the straight line,

$$R(N) = a - bN.$$

This gives us the logistic equation,

$$\frac{dN}{dt} = N(a - bN).$$

Typically, the logistic equation is written as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$

where r is the intrinsic growth rate (the maximal growth rate assuming no environmental resistance) and K is the carrying capacity (the maximum sustainable population).

Equation (1) is also separable, although its solution is somewhat more complicated.

1.3.3 What About Extinction?

We can modify the growth rate to take into account the possibility of extinction of the species if the population decreases below some minimal amount, θ. Thus, $R(N)$ must be negative below θ. The simplest function satisfying this property and the ones above is a quadratic function, yielding the model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{\theta} - 1\right), \tag{2}$$

where $0 < \theta < K$.

2 Slope Fields and Qualitative Assessments of Autonomous First-Order ODEs

An autonomous first-order differential equation is an equation of the form

$$y' = f(y).$$

In other words, t does not explicitly appear in the equation. All of the examples given in Section 1 are modeled by autonomous equations. Autonomous equations may be analyzed in terms of equilibrium points (also known as stationary points). All of the prior examples, except the extinction model, may also be solved analytically, and we will discuss analytic solutions a bit later.

First, we will define equilibrium points and then we will analyze them.

Definition A point y_e is an **equilibrium point** of the autonomous equation $y' = f(y)$ if $f(y_e) = 0$.

We would like to analyze the stability of an equilibrium point. An equilibrium point y_e is **stable** if whenever $y(0)$ is sufficiently close to y_e, then $y(t) \to y_e$ as $t \to \infty$. If there are initial conditions $y(0)$ for which $y(t)$ diverges from y_e, then y_e is **unstable**.

One way to analyze stability is to analyze the slope field of the differential equation. The slope field of a differential equation is the set of all segments with the correct slope through each point in the solution plane. This can be done easiliy in Maple for any differential equation using the *DEplot* command found in the *DEtools* library. We can use *DEplot* to plot the slope field and possible solution curves for a differential equation. We will do this in the context of analyzing the equilibrium points and their stability for the logistic equation.

Example: Logistic Equation

Given the logistic equation (1), reproduced here for simplicity,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

the equilibrium points are determined by setting the derivative equal to zero and solving:

$$0 = rN\left(1 - \frac{N}{K}\right),$$

which gives us as equilibrium solutions $N = 0$, $N = K$. Now, we wish to analyze the stability of these equilibria. For this, we will turn to the Maple worksheet. For the Maple analysis, we will set $r = 2$ and $K = 10$. We first graph $F(N)$ vs. N. From this graph, we see that for $0 < N < K$, $F(N) > 0$ so N is increasing, and for $N > K$, $F(N) < 0$ so N is decreasing. Therefore, $N = K$ is stable, and $N = 0$ is unstable. We will also look at a qualitative analysis of the solution. We can easily sketch some solution curves by hand. We will also check out the slope field of the solution using Maple. You will want to notice that there is a point of inflection at $N = \frac{K}{2}$, which makes sense since that is the point at which $F(N)$ attains its maximum. Note that the slope field also demonstrates that $N = K$ is stable and $N = 0$ is unstable, because all nonnegative solutions tend towards $N = K$ as $t \to \infty$ and all nonnegative solutions tend away from $N = 0$.

3 Analytical Solution to First-Order ODEs

3.1 Separable Equations

Definition A first order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}.$$

A separable equation can be solved by multiplying both sides of the equation by $g(y)$ and integrating over t.

Example: Find a general solution of

$$\frac{dy}{dt} = \frac{y}{1+t^2}.$$

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int \frac{1}{1+t^2}dt$$
$$\ln|y| = \tan^{-1} t + c$$

(the arbitrary constant on the left-hand side has been combined with the one on the right-hand side)

$$e^{\ln|y|} = e^{\tan^{-1} t + c}$$
$$\boxed{y = ce^{\tan^{-1} t}}$$

3.2 Linear Equations

Definition A first order differential equation is **linear** if it can be written in the form

$$y' + p(t)y = q(t), \tag{3}$$

where $p(t)$ and $q(t)$ can be any function of t.

To solve a linear equation, first define $\mu(t) = e^{\int p(t)dt}$, called an **integrating factor**. If we multiply (3) by $\mu(t)$, we obtain

$$e^{\int p(t)dt}y' + p(t)e^{\int p(t)dt}y = q(t)e^{\int p(t)dt}.$$

Noticing that

$$\begin{aligned}\frac{d}{dt}\left(e^{\int p(t)dt}y\right) &= \frac{d}{dt}\left(e^{\int p(t)dt}\right)y + e^{\int p(t)dt}\frac{dy}{dt}\\ &= p(t)e^{\int p(t)dt}y + e^{\int p(t)dt}y',\end{aligned}$$

we see that we have

$$\frac{d}{dt}\left(e^{\int p(t)dt}y\right) = q(t)e^{\int p(t)dt}.$$

To solve, integrate both sides with respect to t and solve for y.

Example: Find a general solution of

$$(t^2+1)y' + 3ty = 6t.$$

First, we need to divide both sides of the equation by t^2+1 to get the equation in the proper form:

$$y' + \frac{3t}{t^2+1}y = \frac{6t}{t^2+1}.$$

Find the integrating factor:

$$p(t) = \frac{3t}{t^2+1} \implies \mu(t) = e^{\int \frac{3t}{t^2+1}dt} = e^{\frac{3}{2}\ln(t^2+1)} = (t^2+1)^{\frac{3}{2}}.$$

Multiply both sides by $\mu(t)$ and solve:

$$\begin{aligned}
(t^2+1)^{\frac{3}{2}}y' + \frac{3t}{t^2+1}(t^2+1)^{\frac{3}{2}}y &= \frac{6t}{t^2+1}(t^2+1)^{\frac{3}{2}} \\
(t^2+1)^{\frac{3}{2}}y' + 3(t^2+1)^{\frac{1}{2}}y &= 6t(t^2+1)^{\frac{1}{2}} \\
\frac{d}{dt}\left((t^2+1)^{\frac{3}{2}}y\right) &= 6t(t^2+1)^{\frac{1}{2}} \\
\int \frac{d}{dt}\left((t^2+1)^{\frac{3}{2}}y\right)dt &= \int 6t(t^2+1)^{\frac{1}{2}}dt \\
(t^2+1)^{\frac{3}{2}}y &= 2(t^2+1)^{\frac{3}{2}} + c \\
&\boxed{y = 2 + c(t^2+1)^{-\frac{3}{2}}}
\end{aligned}$$

3.3 Analytically Solving the Logistic Equation

The logistic equation satisfies the definition for separable equations, so let's solve it.

$$\begin{aligned}
\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) \\
\int \frac{1}{N\left(1-\frac{N}{K}\right)}\frac{dN}{dt}dt &= \int r dt
\end{aligned}$$

The integral on the left-hand side needs to be solved using partial fractions:

$$\begin{aligned}
\int \frac{1}{N\left(1-\frac{N}{K}\right)} &= \int \left(\frac{1}{N} + \frac{\frac{1}{K}}{1-\frac{N}{K}}\right) \\
&= \ln|N| - \ln\left|1 - \frac{N}{K}\right| \\
&= \ln\left|\frac{N}{1-\frac{N}{K}}\right|.
\end{aligned}$$

After some algebra, we obtain

$$N(t) = \frac{cKe^{rt}}{K + ce^{rt}}.$$

For any initial condition, we see that

$$\lim_{t\to\infty} N(t) = K,$$

again verifying that $N = K$ is a stable equilibrium point.

We can also solve this equation in Maple using the *dsolve* function.