Introduction to Modeling *

CURM Background Material, Fall 2014
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1 What Is a Mathematical Model?

A mathematical model is an idealization of a real-world phenomenon, which we use to help us to better understand the physical phenomenon. Although mathematical models have their limitations, due to the necessary simplifying assumptions used to form them, a good model can provide valuable information about a physical system.

1.1 Overview of Modeling

Our goal is to build models describing change in the real world. We wish to solve these models and analyze how god our resulting mathematical explanations and predictions are. In addition, we can classify models based on their mathematical structure.

1.2 Goals of Modeling

Modeling has many goals. First, the modeler wishes to understand

- how a particular system works,
- what causes changes in a system, and
- the sensitivity of a system to certain changes.

In addition, we wish to be able to use models to predict

- what changes will occur in the system, and

*Some of the material in this handout is taken directly from *A First Course in Mathematical Modeling* by Frank Giordano, et al.

- when changes will occur.

For example, one might wish to determine the optimum vaccination scheme to minimize the spread of disease in a rural area. Or, we might wish to analyze the nesting patterns of a particular species of bird in a remote area.

2 Construction of Mathematical Models

Depending on what textbook you look at, modeling may be described as a five- or six-step process. The textbook for infectious disease modeling describes modeling in terms of a seven-step process. The process below is an eight-step process that is a slight modification of that given in the textbook.

So, how do we construct a mathematical model?

Step 1: Identify the Problem. Typically, identification of the problem is a difficult step, because in the real world you are not given a mathematical problem to solve. Usually, you need to sort through large amounts of data and identify a particular aspect of the situation to study. In addition, you need to make sure that the formulation of the problem is sufficiently precise so that it can be translated into a mathematical problem.

Step 2: Identify Relevant Facts about the Problem. For example, in modeling infections, it is important to start identifying key features of the epidemiology of the infection, such as:

- The pre-infectious period (or the latent period) – the time between infection and the onset of symptoms;
- The length of time an individual is infectious;
- The basic reproduction number (to be discussed later);
- Any differentiation in impact of the illness based on age (or other factors).

Step 3: Choose the Type of Modeling Method.

a. Determine the model structure. The type of model chosen will depend on a number of factors. In terms of modeling infectious diseases, the three main considerations are

(1) The natural history of the infection. For example, once someone is infected, do they recover? If so, can they become infected again, or are they now immune (like with chicken pox)?

(2) The accuracy and time period over which model predictions are required. The level of accuracy required influences the type/number of factors that must be included in the model.

(3) The research question.

b. **Select the type of modeling method.** Models may be deterministic or stochastic.

- Deterministic models basically describe what happens "on average" in a population. Parameters are fixed, and not determined by probabilities.
- Stochastic models have at least one parameter that depends on random values.

Step 4: Make Simplifying Assumptions. Due to the complicated nature of physical systems, it is unreasonable to expect to capture all of the factors influencing a system in a mathematical model that is tractable. As such, the number of factors that are considered must be reduced (e.g., by holding certain factors as constant or by neglecting certain factors). Then, relationships among the factors must be determined. Assuming simple relationships may reduce the complexity of a problem.

We can break this down into the following two components.

a. **Specify the Variables and Parameters.** First, we must determine all factors that influence the behavior of the problem identified in Step 1. These factors are our variables and parameters. The variable(s) that the model is trying to explain is (are) the dependent variable(s). The remaining variables are independent variables.

To simplify the modeling process, we may choose to neglect some of the independent variables. We can do this, for example, if the effect of the variable is relatively small compared to other factors affecting the behavior of the problem.

b. **Determine Interrelationships Among the Variables.** Here, again, we may need to make simplifying assumptions because the complexity of the problem may preclude our determining relationships among all of the variables. This is most frequently done using physical or biological principles. Sometimes, we can try to analyze submodels in which we look at one or more of the independent variables individually before connecting the submodels together into one larger model.

Step 5: Construct the Model. Now that we have identified the problem we want to solve (or the question we want to answer), the relevant variables, and relationships among the variables, we use mathematical tools and the relationships between the variables to formulate a model to solve the problem. This model may be formed from smaller submodels that we constructed to analyze the relationships among the variables. In any event, the goal here is to formulate the model in the form of mathematical equations or inequalities that we may analyze.

Step 6: Solve and Interpret the Model. Now that we have formulated a model, we must try to solve it using standard solution procedures. This may involve determining quantitative, qualitative, and/or numerical solutions. At this point, however, the model is often too complicated for us to solve or interpret. In that case, we must go back to Step 2 and try to make more simplifying assumptions.

Step 7: Validate the Model. Once we have solved the model, we must test it to make sure that it (i) reasonably accurately represents the real-world system, and (ii) answers the question. We must also ensure that the model is usable (i.e., that it can be tested by data collection) and makes sense.

The model will be tested by collecting data to which we apply the model. If the model is practically usable and appears to answer the question, then we can claim that the model is reasonable and may implement it. However, since the modeling procedure is not exact, the model typically will need to be re-evaluated and improved.

Step 8: Implement and Maintain the Model. For example, update parameters, etc., as new information becomes available.

3 Modeling Using Proportionality

Recall: $y \propto x$ if and only if $y = kx$ for some constant $k > 0$. Also, there is a transitive rule for proportionality:

$$y \propto x \text{ and } x \propto z \implies y \propto z.$$

Geometrically, the proportionality relationship $y = kx$ represents a line of slope k passing through the origin. In addition, k may be interpreted as the tangent of the angle θ that the line makes with the horizontal axis.

A proportionality relationship may be a reasonable simplifying assumption to a straight-line relationship depending on the size of the y-intercept and the slope of the line, and, perhaps on the domain of the independent variable, as well, since the relative error

$$\frac{y_{\text{true}} - y_{\text{prop}}}{y_{\text{true}}}$$

is greater for smaller values of x.

Examples of proportionality laws:

- Kepler's Third law, devised by Johannes Kepler: $T = cR^{\frac{3}{2}}$, where T is the period (in days) and R is the mean distance to the sun.
- Hooke's law: $F_s = -ky$, where F_s is the restoring force in a spring displaced a distance y from equilibrium.
- Newton's second law: $F = ma$, where a is the acceleration of mass m subject to net external force F.

4 Examples

4.1 Raindrop from a Cloud

Suppose that we are interested in the terminal velocity of a raindrop from a cloud.

Step 1: Identify the Problem. In this case, the problem is more or less clear. We wish to find the terminal velocity of a raindrop from a cloud. We will narrow this down a bit to say that we seek the terminal velocity of a raindrop from a motionless cloud.

Step 2: Identify Relevant Facts about the Problem. We will do this as part of Step 4 below.

Step 3: Choose the Type of Modeling Method. We will use a deterministic model based on Newton's law.

Step 4: Make Simplifying Assumptions. One simplifying assumption was made in the problem statement; namely, that the cloud is motionless. If we draw a free-body diagram, we will see that the only forces acting on the raindrop are gravity (i.e., from the weight of the raindrop), given by F_g, and drag (from air resistance), given by F_d. We will assume that the air resistance is proportional to the product of the surface area S of the raindrop and the square of its speed v. By Newton's second law, we have

$$F = F_d - F_g = ma.$$

Under terminal velocity, $a = 0$, so we Newton's second law gives us $F_g = F_d$.

4.2 Body Weight

Question: How much should an individual weigh?

A rule of thumb given to people who want to run a marathon is 2 lb. of body weight per inch of height. Tables have been designed to suggest weights for different purposes: healthy weights (for doctors); upper weight allowances (organizations concerned about physical conditioning), etc.

In this example, we will examine how height and weight should vary. However, before doing this, we need to take into account the fact that body weight does not just depend on height. For example, bone density could be a factor. Is there a significant variation in bone density from person to person? Is the volume of the bone relatively constant? What about differences (if any) in the density of bone, muscle, and fat?

In this example, we will consider bone density as a constant (by accepting an upper limit) – a simplifying assumption – and discuss how to predict weight as a function of height, gender, age, and body density.

Step 1: Identify the Problem. For various heights, genders, and age groups, determine upper weight limits that represent maximum levels of acceptability based on physical appearance.

Step 2: Identify Relevant Facts about the Problem. We will do this as part of Step 4.

Step 3: Choose the type of Modeling Method. We will use a deterministic model, based on proportionality.

Step 4: Make Simplifying Assumptions. As one simplifying assumption, assume that some parts of the body are composed of an inner and outer core of different densities, and that the inner core is composed primarily of bones and muscle and that the outer core is primarily a fatty material, giving rise to the different densities. Next, we will construct submodels to determine how the weight of each core might vary with height.

First, assume that for adults, certain parts of the body, such as the head, have the same volume and density for different people. So, the weight of an adult is given by

$$W = k_1 + W_{\text{in}} + W_{\text{out}}, \tag{1}$$

where $k_1 > 0$ is the constant weight of those parts having the same volume and density for different individuals and W_{in} and W_{out} are the weights of the inner and outer core, respectively.

We then wish to construct submodels for the weights of the inner and outer core, and then put together these submodels together with Equation (1), we would obtain a model, which we could then solve and verify.