Modeling with Systems of Differential Equations

CURM Background Material, Fall 2014
Dr. Doreen De Leon

1 Introduction

Systems of differential equations may be used to model real-world problems in which interactions occur. Such problems arise in economics, biology, physics, engineering, etc. For example, the interaction of multiple species in an ecosystem may be modeled with a system of two or more first order differential equations. In addition, electrical networks with multiple loops can be modeled by a system of first or second order differential equations.

In some cases, these systems of equations are linear. In this case, they may be solved analytically. However, it is often the case that these systems of differential equations are nonlinear. In that case, they may be analyzed qualitatively through the use of phase portraits and linearization techniques or they may be solved numerically.

2 Examples of Models with Systems of First Order Differential Equations

2.1 Mechanical Vibrations

For example, assume that we have the following situation: mass m_1 is connected to a spring with spring constant k_1, which is connected to a wall on the left; now, assume that mass m_1 has another spring with spring constant k_2 connected to its right side; and attached to the second spring is another mass, mass m_2; and finally, mass m_2 is also connected to a third spring with spring constant k_3 which connects m_2 with the right wall. Determine the motion of the masses.

Step 1: Identify the Problem.

Determine the motion of the masses in the mass-spring system consisting of two masses connected to the wall and to each other by three springs.

Step 2: Identify Relevant Facts about the Problem

Note that:

- the spring with spring constant k_2 connects to both m_1 and m_2;
- mass m_1 is connected to the spring with spring constant k_1 on the left and the spring with spring constant k_2 on the right; and
- mass m_2 is connected to the spring with spring constant k_2 on the left and the spring with spring constant k_3 on the right.

Step 3: Choose the Type of Modeling Method

We will use Hooke's law to model the action of the springs on the masses.

Step 4: Make Simplifying Assumptions.

Assumptions:

- Displacements to the right of equilibrium are positive, and displacements to the left of equilibrium are negative.
- All forces acting to the right are positive forces, and all forces acting to the left are negative forces.
- The spring constants $k_1, k_2,$ and k_3 are all positive and may or may not be the same value.
- The surface is frictionless (so there is no damping).

Variables:

- $m_1, m_2 =$ the masses attached to the springs
- $x_1 =$ the displacement of mass m_1 from its equilibrium position
- $x_2 =$ the displacement of mass m_2 from its equilibrium position
- $k_1, k_2, k_3 =$ spring constant for springs 1, 2, and 3

Step 5: Construct the Model.

The end springs (springs 1 and 3) will behave as in any mass-spring system. The middle spring, however, is not as straightforward. The displacement of the middle spring is given by $x_2 - x_1$. How will it act?

1. If both masses move the same amount in the same direction, then the middle spring will not have changed length, so its displacement $x_2 - x_1 = 0$.
2. If both masses move in the positive direction then the sign of $x_2 - x_1$ will tell us which has moved more. If m_1 moves more than m_2 then the middle spring is being compressed, and $x_2 - x_1 < 0$. Or, if m_2 moves more than m_1, then the spring is being stretched and $x_2 - x_1 > 0$.

3. If both masses move in the negative direction, the behavior will be opposite of the behavior in 2. In other words, if m_1 moves more than m_2, then the spring is being stretched and $x_2 - x_1 > 0$; and if m_2 moves more than m_1, then the spring is being compressed and $x_2 - x_1 < 0$.

4. If m_1 moves in the positive direction and m_2 moves in the negative direction, then the spring is compressed and $x_2 - x_1 < 0$.

5. If m_2 moves in the positive direction and m_1 moves in the negative direction, then the spring is stretched and $x_2 - x_1 > 0$.

A free body diagram on each of the masses will gives us the system of differential equations:

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \frac{d^2 x_2}{dt^2} &= -k_2(x_2 - x_1) - k_3 x_2, \end{aligned}$$

or

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 \frac{d^2 x_2}{dt^2} &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned}$$

We will skip Step 6: Solve and Interpret the Model at this time.

2.2 Population Models

The simplest model for the interaction between two populations is given by a system of differential equations. The simplest such model describing a two-species, predator-prey relationship is the Lotka-Volterra model. The key to this model is the use of the mass action principle to model the interaction of the two species (i.e., the degree of interaction is proportional to the product of the populations of the two species). The model is given by

$$\frac{dx}{dt} = rx - axy \tag{1}$$
$$\frac{dy}{dt} = -my + bxy, \tag{2}$$

where x represents the prey and y represents the predator. In addition to describing predator-prey systems, the system given by (1), (2) describes a host-parasite interaction. In general, a system of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + bxy \\ \frac{dy}{dt} &= cy + dxy \end{aligned}$$

can be used to model any two-species ecosystem. Changing the signs on the terms (and/or eliminating interaction terms) will yield models for other types of biological interactions, e.g., mutualism (both species benefit), commensalism (one species benefits, the other is unaffected), competition, etc. For example, a commensal system would look like:

$$\begin{aligned}\frac{dx}{dt} &= rx \\ \frac{dy}{dt} &= by + cxy,\end{aligned}$$

where x is the unaffected species (thus, the lack of an interaction term) and y is the benefited species.

2.3 Diffusion through a Membrane

The diffusion of a substance (such as glucose, potassium, or salt) in a medium (such as blood or water) can be modeled by a system of first-order linear ordinary differential equations.

2.3.1 Single-walled membrane

Suppose there are two solutions of a substance separated by a membrane of permeability P. Assume that the amount of substance that passes through the membrane at any given time is proportional to the difference in the concentrations of the substance. Let x_1 and x_2 be the concentration of the solution on the left and right sides of the membrane, respectively, and V_1 and V_2 be the volume of solution on the left and right sides of the membrane, respectively. Then the rate of change in concentration on each side of the membrane is given by the system of equations

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{P}{V_1}\left(x_2 - x_1\right) \\ \frac{dx_2}{dt} &= \frac{P}{V_2}\left(x_1 - x_2\right),\end{aligned}$$

where the initial amounts of x_1 and x_2 are given.

2.3.2 Double-walled membrane

Now, consider diffusion through a double-walled membrane, with inner wall of permeability P_1 and outer wall with permeability P_2, where $0 < P_1 < P_2$. Let x be the concentration of the solution inside the inner wall and y be the concentration of solution between the two walls. Let V_1 be the volume of solution inside the inner wall and V_2 be the volume of solution between the two walls. Finally, let C be the (constant) concentration of the solution outside the outer wall. Then the rate of change in concentrations of solution x and y is given by the system of equations

$$\frac{dx}{dt} = \frac{P_1}{V_1}(y - x)$$
$$\frac{dy}{dt} = \frac{1}{V_2}(P_2(C - y) + P_1(x - y)).$$

3 The Eigenvalue Method for Determining Stability

First, we will define what we mean by equilibrium solution and stability. The point $\mathbf{x}_e = (x_e, y_e)$ is an equilibrium solution of the autonomous system of equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

if $f(x_e, y_e) = 0$ and $g(x_e, y_e) = 0$. This definition generalizes to systems of more equations in more unknowns.

Definition

- $\mathbf{x}_e$ is **stable** if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x}(0) - \mathbf{x}_e| < \delta$, then $|\mathbf{x}(t) - \mathbf{x}_e| < \epsilon$ for all t. In other words, $\mathbf{x}_e$ is stable if every solutions that starts near the equlibrium solution stays near the equilibrium solution.
- $\mathbf{x}_e$ is **asymptotically stable** if it is stable and there exists $\delta > 0$ such that if $|\mathbf{x}(0) - \mathbf{x}_e| < \delta$, then $\mathbf{x}(t) \to \mathbf{x}_e$ as $t \to \infty$. In other words, the equilibrium solution is asymptotically stable if every solution that starts near the equilbrium solution converges to the equilibrium solution as $t \to \infty$.
- If $\mathbf{x}_e$ is not stable, it is called **unstable**.

3.1 Linear Systems

Suppose we have the system of differential equations given by

$$\frac{dx}{dt} = ax + by \tag{3}$$
$$\frac{dy}{dt} = cx + dy. \tag{4}$$

It is easy to verify that the only equilibrium solution to (3), (4) is (0, 0).

Now, rewrite the system given in (3), (4) in matrix form:

$$\frac{d}{dx}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We will first find the eigenvalues of the coefficient matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\begin{aligned} 0 &= det(A - \lambda I) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda - bc. \end{aligned}$$

Therefore,

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 + 4bc}}{2}.$$

Analysis of the solution gives us the following result.

- (0, 0) is asymptotically stable if Re $\lambda_1 < 0$ and Re $\lambda_2 < 0$.
- (0, 0) is stable if Re $\lambda_1 =$ Re $\lambda_2 = 0$ (i.e., λ_1 and λ_2 are purely imaginary).
- (0, 0) is unstable if Re $\lambda_1 > 0$ or Re $\lambda_2 > 0$.

3.2 Nonlinear Systems

In this case, stability is determined by linearization. Given the autonomous system of equations

$$\frac{dx}{dt} = f(x,y) \tag{5}$$

$$\frac{dy}{dt} = g(x,y), \tag{6}$$

first start by determining the Taylor series expansion of $f(x,y)$ and $g(x,y)$ about the equilibrium solution $\mathbf{x}_e = (x_e, y_e)$. This gives

$$\begin{aligned} f(x,y) = f(x_e,y_e) &+ \frac{\partial f}{\partial x}(x-x_e) + \frac{\partial f}{\partial y}(y-y_e) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-x_e)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y-y_e)^2 \\ &+ \frac{\partial^2 f}{\partial x \partial y}(x-x_e)(y-y_e) + \ldots \\ g(x,y) = g(x_e,y_e) &+ \frac{\partial g}{\partial x}(x-x_e) + \frac{\partial g}{\partial y}(y-y_e) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x-x_e)^2 + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(y-y_e)^2 \\ &+ \frac{\partial^2 g}{\partial x \partial y}(x-x_e)(y-y_e) + \ldots. \end{aligned}$$

If $f(x, y)$ and $g(x, y)$ are sufficiently "nice," then we can neglect all higher order terms, writing

$$f(x, y) \approx f(x_e, y_e) + \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial y}(y - y_e) = \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial y}(y - y_e)$$
$$g(x, y) \approx g(x_e, y_e) + \frac{\partial g}{\partial x}(x - x_e) + \frac{\partial g}{\partial y}(y - y_e) = \frac{\partial g}{\partial x}(x - x_e) + \frac{\partial g}{\partial y}(y - y_e).$$

We can rewrite this in matrix form as

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} (x - x_e) \\ (y - y_e) \end{pmatrix}.$$

If we define

$$\mathbf{z} = \begin{pmatrix} x - x_e \\ y - y_e \end{pmatrix},$$

then the system of equations in (5), (6) may be approximated by the linear system

$$\frac{d\mathbf{z}}{dt} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mathbf{z}.$$

Note:

1. The coefficient matrix is the Jacobian of the function $\mathbf{F}(x, y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$, J, evaluated at the equilibrium solution (x_e, y_e).
2. Since the system is linear, the stability of its equilibrium solution, (0, 0), may be determined using eigenvalue theory.

What information does the nature of the eigenvalues of the linearized system give about the nonlinear system?

Theorem 1. *If $\mathbf{z}' = J(x_e, y_e)\mathbf{z}$ represents the linearization of the system (5), (6) with equilibrium solution (x_e, y_e) then the following hold.*

- *If (0, 0) is asymptotically stable, then (x_e, y_e) is asymptotically stable.*
- *If (0, 0) is unstable, then the equilibrium solution of (x_e, y_e) is unstable.*
- *If (0, 0) is stable, but not asymptotically stable, then no information is given about the stability of (x_e, y_e).*

Why do we have no information about the stability of (x_e, y_e) if the linearized system has an equilibrium solution that is stable, but not asymptotically stable? The reason is that a perturbation might result in eigenvalues for which one has a positive real part.

3.3 Example: Hardwood and Softwood Trees

In an unmanaged tract of forest, hardwood and softwood trees compete for available land and water. Hardwood trees grow more slowly, but are more durable and produce more valuable timber than softwood trees. Softwood trees grow quickly and consume the available water and nutrients. Hardwood trees grow taller than softwood trees, shading new seedlings, and are more resistant to disease. Assume that in isolation, the hardwood trees have a growth rate of 10% and the softwood trees have a growth rate of 25%. Also assume that an acre of forest land can support approximately 10,000 tons of hardwood trees or 6,000 tons of softwood trees. The extent of competition has not be numerically determined. Can these two types of trees coexist on the one tract indefinitely?

Step 1: Identify the Problem.

Determine under what conditions the hardwood and softwood trees can coexist.

Step 2: Idenfity Relevant Facts about the Problem.

Note that:

- hardwood trees grow more slowly than softwood trees;
- hardwood trees are more durable than softwood trees;
- hardwood trees grow taller than softwood trees;
- hardwood trees are more resistant to disease than softwood trees.

Step 3: Choose the Type of Modeling Method.

We will use a deterministic competition model, assuming logistic growth if the species grows in isolation.

Step 4: Make Assumptions.

- Variables
  - H = population of hardwood trees (tons/acre)
  - S = population of softood trees (tons/acre)
  - g_H, g_s = growth rate for hardwoods, softwoods (tons/acre/year)
  - c_H, c_S = loss due to competition (tons/acre/year)
  - b_1, b_2 = interaction coefficients for hardwoods, softwoods (per ton per acre per year)
  - r_H, r_S = intrinsic growth rate for hardwoods, softwoods (tons/acre/year)
  - K_H, K_S = carrying capacity of hardwoods, softwoods (tons/acre)
- Assumptions

- $r_H = 0.10, r_S = 0.25$
- $K_H = 10000, K_S = 60000$
- $g_H = r_H H \left(1 - \dfrac{H}{K_H}\right)$
- $g_S = r_S S \left(1 - \dfrac{S}{K_S}\right)$
- $c_H = b_1 SH, c_S = b_2 SH$
- $b_1, b_2 > 0$

Step 5: Construct the Model.

The net rates of change in the populations of the hardwood and softwood trees are given by

$$\frac{dH}{dt} = g_H - c_H \text{ and } \frac{dS}{dt} = g_S - c_S.$$

This leads to the system of equations

$$\frac{dH}{dt} = 0.10H\left(1 - \frac{H}{10000}\right) - b_1 SH \tag{7}$$

$$\frac{dS}{dt} = 0.25S\left(1 - \frac{S}{6000}\right) - b_2 SH \tag{8}$$

The goal is to determine the nonzero equilibrium solution(s) and the conditions for the stability of the equilibrium solution(s).

Step 6: Solve and Interpret the Model.

Using Maple, we determine that there is only one equilibrium solution for which both populations are nonzero. The equilibrium solutions are

$$H_e = \frac{10000 - 6 \cdot 10^8 b_1}{1 - 2.4 \cdot 10^9 b_1 b_2} \text{ and } S_e = \frac{6000 - 2.4 \cdot 10^8 b_2}{1 - 2.4 \cdot 10^9 b_1 b_2}. \tag{9}$$

For these solutions to be positive, we require that

$$\begin{aligned} -10000 + 6 \cdot 10^8 b_1 &> 0 \\ -6000 + 2.4 \cdot 10^8 b_2 &> 0. \\ 1 - 2.4 \cdot 10^9 b_1 b_2 &> 0 \end{aligned}$$

The first equation is true if $b_1 < \dfrac{1}{60000}$ and the second equation is true if $b_2 < \dfrac{1}{40000}$. Since $1 - 2.4 \cdot 10^9 \dfrac{1}{60000} 0.000025 = 0$, the third equation is also true if $b_1 < \dfrac{1}{60000}$ and $b_2 < \dfrac{1}{40000}$.

To analyze the stability of the equilibrium solution in (9), first find the Jacobian of $\begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$, where $F(x,y) = 0.10H\left(1 - \dfrac{H}{10000}\right) - b_1 SH$ and $G(x,y) = 0.25S\left(1 - \dfrac{S}{6000}\right) - b_2 SH$.

From Maple, the Jacobian is given by

$$A = \begin{pmatrix} \frac{6000b_1 - 0.1}{1 - 2.4 \cdot 10^9 b_1 b_2} & \frac{6 \cdot 10^8 b_1^2 - 10000 b_1}{1 - 2.4 \cdot 10^9 b_1 b_2} \\ \frac{2.4 \cdot 10^8 b_2^2 - 6000}{1 - 2.4 \cdot 10^9 b_1 b_2} & \frac{10000 b_2 - 0.25}{1 - 2.4 \cdot 10^9 b_1 b_2} \end{pmatrix}.$$

Analyzing the eigenvalues of this matrix using Maple gives a further restriction on b_1 and b_2 in order for the equilibrium to be asymptotically stable. The additional condition is $b_2 < -0.6b_1 + 0.000035$.

Step 7: Validate the Model.

To validate the model, we will assume that $b_1 = \dfrac{t}{60000}$ and $b_2 = \dfrac{t}{40000}$, where $0 < t < 1$. From Maple, we see that these values for b_1 and b_2 satisfy all conditions for stability. We will verify this by again determining the equilibrium solutions and verifying the stability of the nonzero solution.

We see from the work in Maple that, assuming $t < 1$, the equilibrium solution

$$H_e = \frac{10000}{t+1} \text{ and } S_e = \frac{6000}{t+1}$$

is asymptotically stable.

4 Phase Portraits

Given an autonomous system of equations, $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{F}$ has continuous first partial derivitves, a solution to $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is a set of parametric equations

$$\begin{aligned} x_1 &= x_1(t) \\ x_2 &= x_2(t) \\ &\vdots \\ x_n &= x_n(t) \end{aligned}$$

The solution curve whose coordinates are $(x_i(t), x_j(t)), i \neq j$ is called a trajectory of the system. The $x_i x_j-$plane is called the **phase plane**. A graph of trajectories in the phase plane is called a **phase portrait**.

The following give some properties of phase portraits.

1. There is at most one trajectory through any point in the phase plane (due to uniqueness).
2. A trajectory that starts at a point other than a rest point cannot reach a rest point in a finite amount of time. (In the case of a system of two equations, a rest point is an equilibrium point).

3. No trajctory can cross itself unless it is a closed curve. If it is a closed curve, it is a periodic solution.

There are two ways to do a phase portrait in Maple. One is using the *DEplot* command, and the other is to use the *phaseportrait* command. Since we have been using *DEplot* in this class to this point, we will continue to do so.

Example: Commensalism

Given the following system of first-order differential equations modeling a commensal relationship between two species, we wish to determine the equilibrium solutions and their stability using phase portraits.

$$\frac{dh}{dt} = 0.5h\left(1 - \frac{h}{100}\right)$$
$$\frac{ds}{dt} = 0.01s\left(1 - \frac{s}{25}\right) + 0.002sh.$$

Note: h is the population size of the "host" species, and s is the popoulation size of the secondary species.

The equilibrium solutions for this system are found by setting the derivatives equal to zero and solving the resulting system of equations:

$$0 = 0.5h\left(1 - \frac{h}{100}\right)$$
$$0 = 0.01s\left(1 - \frac{s}{25}\right) + 0.002sh.$$

which gives: $(h_e, s_e) = (0, 0), (0, 25), (100, 0), (100, 525)$ as equilibrium solutions. Look at the phase portrait and analyze the stability of these equilibrium solutions. Based on the phase portrait, $(h_e, s_e) = (100, 525)$ is asymptotically stable and all other equilibrium solutions are unstable. This means that both species are expected to coexist indefinitely, which is to be expected due to the nature of the commensal relationship.