**1.2 FERMI QUESTIONS—ORDER OF MAGNITUDE.** Fermi questions were named after Enrico Fermi, the great physicist who contributed to both experiments and theory concerned with atomic nuclei and fundamental particles. (He was a master at computing approximations to answers when it seemed that no information was available.) The point of such questions is that reasonable assumptions linked with simple calculations can often narrow down the range of values within which an answer must lie. The *order of magnitude* refers to the power of 10 of the number that fits the value. To increase an order of magnitude means to increase by a factor of 10. Very often an order of magnitude calculation is all that the interest in a problem justifies. Even if more precision is needed, an order of magnitude calculation done firs: may indicate whether or not it is worthwhile to pursue the problem, and sometimes may indicate how the next approximation to the required value can best be obtained.

Here are some Fermi questions:

**A.** How many golf balls will fit in a suitcase? Assume that the suitcase is $30'' \times 8'' \times 24''$. The volume is $30'' \times 8'' \times (100/4)'' \approx 6 \times 10^3$ in$^3$. Assume that each golf ball is a sphere 1 in. in diameter. The volume of the ball is a little less than 1 in$^3$. The order of magnitude of the number of golf balls that will fit in a suitcase is $10^4$.

This question could not be taken too seriously unless it were asked by a traveling golf ball salesman. Since the size of the suitcase is not specified, there is no point going to the closet to measure a real one. Imagine a reasonable size. Nor is there any point in worrying about whether or not the balls are close packed as nested spheres; the packing factor could not be greater than 1.5. Surely the diameter of a golf ball is closer to 1 in. than 2 in. Doubling the diameter would increase the ball volume by a factor of 8. That would reduce the number of balls in the suitcase by an order of magnitude. Consider the reasonableness of the order of magnitude answer. Surely the number of golf balls that would fit in a standard suitcase must be greater than 1000 and less than 100,000. Our answer is good within a factor of 10.

**B.** How many piano tuners are there in New York City? Assume $10^7$ people in New York and $2 \times 10^6$ families. Assume 1 piano for every 5 families; therefore, $4 \times 10^5$ pianos. Assume each piano tuned once every 2 years; therefore, $2 \times 10^5$ pianos tuned each year. Assume each tuner tunes 2 a day for 250 days a year. (At $10 per tuning, he barely makes a living—a factor of 2 could make a big difference to the tuner.) Therefore, ($20 \times 10^4$ tunings per year)/(500 tunings per year per tuner) = 400 tuners.

It is unlikely that New York City has less than 40 piano tuners or more than 4000. If you do not like the assumptions made, choose your own reasonable guesses and see if your answer is not of the same order of magnitude.



Note that sometimes in these calculations one significant figure is carried along. The extent to which you do this depends on the problem and your style; rules would be cumbersome and probably useless. For instance, whether 400 is of order of magnitude $10^2$ or $10^3$ is a silly question, because a reasonable answer depends on the meaning of the number and the way it is going to be used. When in doubt, carry an extra figure along. Note, incidentally, that a factor of 2 in one of the assumptions makes a big difference to the piano tuners but not to the final result of our order of magnitude calculation.

**C.** How many cells are there in a human body? Assume that the average cell diameter is 10 microns ($\mu$) = $10^{-5}$ meter (m). Then, volume = $10^{-15}$ m$^3$. The order of magnitude of human volume is $10^{-1}$ m$^3$. Therefore, there are $10^{14}$ cells in a human body.

This question illustrates again how some information can be obtained out of very little definite knowledge. Living cells come in a great range of sizes. However, they can all be seen with an ordinary light microscope and therefore must have a diameter larger than the wavelength of light. They can scarcely be seen with the unaided eye and so must have a diameter smaller than 0.1 millimeter (mm). We assumed that the diameter was the geometric mean between these values. (The geometric mean of $A$ and $B$ is $\sqrt{A \times B}$. In this case, it is $\sqrt{(10^{-6})(10^{-4})} = 10^{-5}$. The arithmetic mean would be practically the same as $10^{-4}$.) Notice that for these calculations it makes no sense to differentiate between the volume of a sphere and that of a cube. The volume of the human body could be estimated by assuming a reasonable height, width, and thickness of a column that is human size. An alternative method requires knowing that 1 liter of water has a mass of 1 kilogram (kg) and a volume of $1 \times 10^{-3}$ m$^3$. A cubic meter of water (or flesh) would therefore have a mass of 1000 kg and would weigh about a ton (1 kg weighs 2.2 pounds; therefore 1000 kg weighs 2200 pounds or 1.1 tons). The assumed volume for the body was 1/10 m$^3$.

### SAMPLES

1. How many hairs are on a human head? (Assume that the spacing between hairs is 1 mm. Then there are 10/centimeter (cm) along a line or 100/cm$^2$. We figure $2 \times 10^4$ hairs on a human head. Do your assumptions lead to results of the same order of magnitude?)

2. How many individual frames of film are needed for a feature length film? (We get $1.5 \times 10^5$.)

3. What is the ratio of spacing between gas molecules to molecular diameter in a gas at standard temperature and pressure? [A mole ($6 \times 10^{23}$) of gas molecules at STP occupies 22.4 liters. A molecular diameter might be $2 \times 10^{-8}$ cm. Compare available volume per molecule with the volume of a molecule. We get (spacing between molecules)/(molecular diameter) = 10.]

## 1.7 ACCURACY, PRECISION, AND COMMON SENSE

**A.** Does an average of 10 readings, each with $n$ significant figures, yield a value with $n + 1$ significant figures?

$$\bar{x} = \frac{\sum_i x_i}{n} = \frac{18.4 + 19.2 + 18.5 + 18.9 + 18.8 + 18.5 + 18.9 + 18.6 + 19.0 + 18.7}{10} \stackrel{?}{=} 18.75$$

It is commonly assumed that this is the case. The standard argument is that since the first figure after the decimal is significant for each reading, the first figure after the decimal should be significant for the sum (in this case, 187.5). Dividing by 10 does not change this significance, and therefore the average has an additional significant figure.

This situation illustrates the difficulty of laying down hard and fast rules about the analysis of data. In the example given, it may be that the experimenter could judge each reading within ±0.05, but for one reason or another the average value did not always lie within that range. The spread of readings indicates that the figure after the decimal point was not significant in the original technical sense. Circumstances such as this are common in actual experiments. In general, there is no justification for citing an additional significant figure for an average value.

**B.** The error in a student experiment is *not* the difference between the student's experimental value and the textbook value. Error limits of individual measurements should be determined and cited by the experimenter. The compound error due to errors in the individual variables should then be calculated. If the error limits of the experimental value do not overlap the textbook value, there is evident reason to reassess the original judgments of possible error limits or to look for a mistake. An experimental value of 84 ± 20 is not in disagreement with a textbook value of 100. (Of course, if the assignment was to determine the value within 5 per cent, the experimenter has other problems.)

**C.** For a compound value made up of several individual values, do not seek more precision for any one value than is justified by the precision with which you know the others. If one value has a 10 per cent error, there is usually no point in obtaining another value to within 1 per cent. Notice especially how unimportant small errors become if they express standard deviations, linked with others through the square root of the sum of their squares. (In this case, the product of two values, one with a 10 per cent error and the other with a 3 per cent error, has an error of $\sqrt{10^2 + 3^2} = 10.4$ per cent, and not 13 per cent)

If the diameter of a cylinder is measured to within 2 per cent and the height to within 1 per cent, then the volume is known to within 5 per cent,



assuming that the data were not sufficient to justify statistical procedures. The use of three significant figures for $\pi$ (3.14) yields $\frac{1}{6}$ per cent error margins, which do not add to the overall error margins for the volume. The use of four significant figures would not be justified. There would also be no point in improving the precision of the measurement of the height, since the major contribution to the error is provided by the diameter measurement.

**D.** You have no doubt heard that ancient saw, "If a thing is worth doing, it is worth doing well." That, of course, is nonsense. If a thing is worth doing, it is worth doing *well enough for the purpose at hand.* To do it any better than that is surely silly and probably wrong. Do not think, however, that this realistic view makes life easier. The purpose at hand may require years of devoted and meticulous work. Furthermore, the individual is faced with the awful responsibility of using his head to determine the requirements of the problem. No rules exist.

Precision is expensive. In 1954 a particular cross-section value for a high-energy particle reaction was measured during the course of an afternoon to an accuracy of 15 or 20 per cent. During the following two years, three scientists and numerous technicians spent a considerable fraction of their time and over $100,000 to obtain that cross section to a ±4.4 per cent error. There was good reason for obtaining that precision, and the probable cost and difficulty were carefully considered in advance. The very first response that anyone should make when faced with a task is: "For what purpose is this required?" Your procedure will often depend strongly on the answer.

What is the area of your front yard? Your choice of measuring instruments depends on the purpose for which the information is required. If you want to know the area so that you can buy lime in 80-lb bags at the store, then you can measure the yard by looking at it. Lime is cheap, you cannot buy a fractional bag, and the exact dosage is unimportant. If you want to know the area in order to buy grass seed at $1.00/lb, then you should pace out the yard and perhaps even use pencil and paper to check your arithmetic. Whether the yard is exactly rectangular, or whether your pace is 5 feet (ft) or 5 ft 3 in is unimportant. You certainly would not use a meter stick. (Where is the seed salesman who can advise you in terms of lb/m²?) If you want to measure the area of your yard for legal purposes, the assessor will probably insist on knowing the area to within 0.01 acre. A surveyor's transit is the appropriate measuring instrument.

## 1.8 PROPER CITATION OF ERROR LIMITS

See Appendix 11 on p. 256.

SCIENCE TOOLBOX

# Estimation

Many important calculations in science can be done by estimation. Estimation gives us a quick way to gain insight by using rough or approximate numbers. This is especially appropriate in astronomy, where many measurements have low precision and few important numbers are known to more than two or three significant figures. In estimation, we are satisfied with a precision of a factor of 2 or even a factor of 10, which is known as an *order of magnitude.* Estimation is perhaps the best way to come to grips with the enormous range of scales in the universe.

To start with a frivolous example, how many pieces of paper would you have to pile up to reach the Moon? A ream of 500 pages of writing paper is roughly 40 millimeters (mm) thick. Although we might be able to make a more precise measurement of 39 mm, or 39.6 mm, in estimation we make do with the round number 40. The thickness of a single sheet is therefore

$$\frac{40}{500} = 0.080 \text{ mm}$$

The distance to the Moon is 384,000 km. Expressed in millimeters, this is 384,000 times 1,000,000, or $3.84 \times 10^{11}$ mm. So the number of pieces of paper required to reach the Moon is the distance divided by the thickness of a single page:

$$\frac{3.84 \times 10^{11}}{0.080} = 4.8 \times 10^{12}$$

This number, nearly 5 trillion, is about the same order of magnitude as the sum of all the pages in all the world's books. Or imagine that the pieces of paper are instead dollar bills. The bank balance of most people would be a pile smaller than the height of a person. However, the worth of the richest person in the world would be a pile that reached several times around Earth, and the United States budget would be a pile that reached nearly halfway to the Moon!

You could also easily estimate how many times Earth would fit inside the largest planet in the solar system, Jupiter. Earth has a diameter of roughly 13,000 km, and Jupiter has a diameter of roughly 140,000 km. The volume of Jupiter, in cubic kilometers, is

$$\begin{aligned}&\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\\ &= \frac{4}{3} \times 3.14 \times (70{,}000)^3\\ &= 1.4 \times 10^{15} \text{ km}^3\end{aligned}$$

If we imagine packing Jupiter with many versions of Earth, such that they just touch one another like marbles in a jar, then each Earth will take up a space equal to a cube that just nestles around a sphere. (This is an approximation; marbles in a jar will actually pack more tightly than this. Try it!) The volume of each cube surrounding Earth is $D^3$, or $(13{,}000)^3 = 2.2 \times 10^{12}$ km$^3$. So the number of times Earth will fit into Jupiter is given by

$$\frac{1.4 \times 10^{15}}{2.2 \times 10^{12}} \approx 600$$

Scientists use a wavy equals sign ($\approx$) or another similar symbol ($\sim$ or $\cong$) to mean approximately equals or roughly equals. Jupiter dwarfs Earth, and we will see later that the largest storms on Jupiter are even bigger than Earth.

The last example will give us a sense of the vast distance between stars. The Pioneer 11 spacecraft left the solar system several years ago and is traveling at about 110,000 kilometers per hour (km/h). How long will it take to reach the distance of the nearest stars? Alpha Centauri is 1.3 parsecs (pc), or $3.9 \times 10^{13}$ km, away (the parsec distance unit will be introduced later in the book). So the number of hours that Pioneer 11 will take to reach the distance of the nearest star is

$$\frac{3.9 \times 10^{13}}{110{,}000} = 3.5 \times 10^8 \text{ h}$$

We can convert this to $(3.5 \times 10^8) / (24 \times 365) \approx 40{,}000$ years. This is a sobering reflection on the capabilities of today's spacecraft. We will need new technologies before we can explore the stars.

Scientists try to combine only numbers that have similar precision (see Appendix A-4 for a discussion of precision). Why? Because the result is governed by the number with the lowest precision, or the least number of significant figures. In other words, combining a great measurement with a lousy measurement will give you a lousy result. We can see this in our first example. Suppose that we know the distance to the Moon with a precision of eight significant figures or an accuracy of about 1 cm. (In fact we do, using radar measurements!) Yet our measurement of the thickness of a piece of paper is only good enough to have two significant figures. This means that our estimate of the number of pieces of paper to reach the Moon should only be quoted to two significant figures—it is limited by our least precise measurement.

The way to get good at estimation is to practice. Try it! Just remember to express all the quantities you are combining in the same units. Mixing meters and kilometers or grams and kilograms is the easiest way to make a mistake that will throw your answer way off. If you are using a calculator, be careful to enter very large or very small numbers correctly in scientific notation (see Appendix A-1 for tips). And if you want to estimate to only one or two significant figures, you do not even need a calculator! This is what scientists mean by a "back of the envelope" calculation.