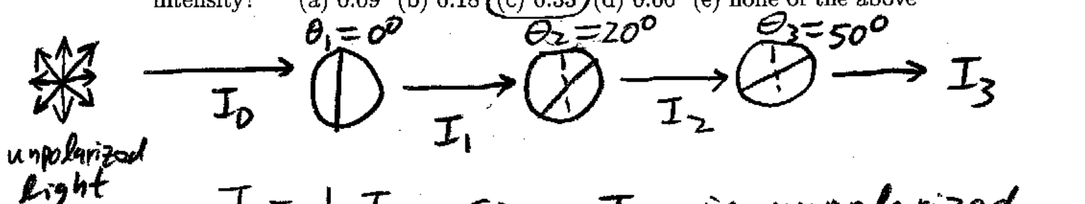
Physics 4C Mid-Term Exam #2

Instructions: Read the problems *carefully* and give the best answer, based on the material presented in class and in the text. The multiple-choice questions are worth 10 points each.

(1) A beam of unpolarized light is incident on three polarizers. The first polarizer has its polarization axis vertical. The second polarizer has its axis 20 degrees to vertical. The third polarizer has its axis 50 degrees from vertical. What is the ratio (I/I_0) of the final intensity of the transmitted light to the original incident intensity? (a) 0.09 (b) 0.18 (c) 0.33 (d) 0.66 (e) none of the above



Find $- I_3/I_0$

$I_1 = \frac{1}{2} I_0$, since I_0 is unpolarized.

$I_2 = I_1 \cos^2(20^\circ - 0^\circ) = I_1 \cos^2(20^\circ)$

$I_3 = I_2 \cos^2(50^\circ - 20^\circ) = I_2 \cos^2(30^\circ)$

so: $\frac{I_3}{I_0} = \frac{1}{2}\cos^2(20^\circ)\cos^2(30^\circ) = \boxed{\frac{I_3}{I_0} = 0.33}$

→ choice (c)

(2) The Compton wavelength of the electron is equal to the shift in wavelength due to Compton scattering (in other words, $\Delta\lambda = \lambda' - \lambda_0$), when the scattering angle for the photon $\theta = 90$ degrees. The electron mass is 9.11×10^{-31} kg = 511 keV/c^2. What is the value of the Compton wavelength of the electron, in meters?
(a) 3.85×10^{-13} (b) 2.43×10^{-12} (c) 5.11×10^{-11} (d) 2.02×10^{-9} (e) 4.42×10^{-9}

The Compton effect –

Find $- \Delta\lambda$

$$\Delta\lambda = \lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta)$$

$$= \frac{h}{m_e c}(1 - \underbrace{\cos 90^\circ}_{=0})$$

$$= \frac{h}{m_e c} = \frac{6.626 \times 10^{-34}\ \text{J-s}}{(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^{8}\ \frac{\text{m}}{\text{s}})}$$

$$\boxed{\Delta\lambda = 2.43 \times 10^{-12}\ \text{m}}$$

→ Choice (b)

(3) A relativistic electron (with $v > 0.1\ c$) has a kinetic energy equal to twice its rest energy. Determine its speed. (a) 0.76 c (b) 0.81 c (c) 0.94 c (d) 0.54 c (e) 0.87 c

Relativistic ($v > 0.1c$) kinetic energy – Find – v of the electron.

$K_{rel} = (\gamma-1)mc^2 = 2m_ec^2$, here.

$$(\gamma-1)\cancel{m_ec^2} = 2\cancel{m_ec^2}$$

$$\gamma - 1 = 2$$

$$\gamma = 3 = \frac{1}{\sqrt{1-(v/c)^2}}$$

$$1-(v/c)^2 = 1/3^2 = 1/9$$

$$(v/c)^2 = 8/9$$

$$v/c = \sqrt{8/9}$$

$$\boxed{v = 0.94c} \boxed{\to \text{choice (c)}}$$

(4) Suppose an electron is moving at a speed of 0.999 999 90 c = 0.99999990 c. (That's seven instances of the numeral "9.") At what speed, expressed as a fraction of the speed of light in empty space c, does a proton have the same linear momentum as this electron? The masses of the electron and the proton are listed in the Physics 4C Formula List, at the end of this exam.
(a) 0.13762 (b) 0.54460 (c) 0.68935 (d) 0.77283 (e) 0.99993

Find – v_p of the proton.

At relativistic speeds (with $v > 0.1c$), $p = \gamma m v$, NOT $p = mv$.

P_p (for the proton) $= \gamma_p m_p v_p = \gamma_e m_e v_e = P_e$ (for the electron), since even in special relativity, linear momentum is conserved.

$$\gamma_p v_p = \left(\frac{m_e}{m_p}\right)\gamma_e v_e = \left(\frac{0.511\ \cancel{MeV/c^2}}{938.272\ \cancel{MeV/c^2}}\right)\frac{(0.999\ 999\ 90\ c)}{\sqrt{1-(0.999\ 999\ 90)^2}}$$

$\gamma_p v_p = 1.2178\ c \equiv Ac$, letting $A = 1.2178$.

$$\frac{v_p}{\sqrt{1-(v_p/c)^2}} = Ac$$

$$\frac{v_p^2}{1-(v_p/c)^2} = A^2c^2$$

$$v_p^2 = A^2c^2 - A^2c^2(v_p/c)^2$$

$$(v_p/c)^2 = A^2 - A^2(v_p/c)^2$$

$$(v_p/c)^2[1+A^2] = A^2$$

$$\left(\frac{v_p}{c}\right)^2 = \frac{A^2}{1+A^2} = \frac{1.483}{1+1.483} = \frac{1.483}{2.483}$$

$$\frac{v_p}{c} = \sqrt{\frac{1.483}{2.483}} = \boxed{\frac{v_p}{c} = 0.77283} \boxed{\to \text{choice (d)}}$$

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(5) Suppose a spaceship is traveling at $v = 0.75$ c, relative to a stationary observer. This spaceship launches a probe at a speed of $u_x' = 0.5$ c, relative to the spaceship, and in the same direction as the motion of the spaceship. What speed does the probe have, relative to the stationary observer?

(a) 3.333 c (b) 1.25 c (c) 0.993 c (d) 0.909 c (e) none of the above

Relativistic velocity addition –

$$u_x = \frac{u_x' + v}{1 + \left(\frac{u_x' v}{c^2}\right)} = \frac{0.5c + 0.75c}{1 + \left[(0.5)(0.75)\frac{\cancel{c^2}}{\cancel{c^2}}\right]}$$

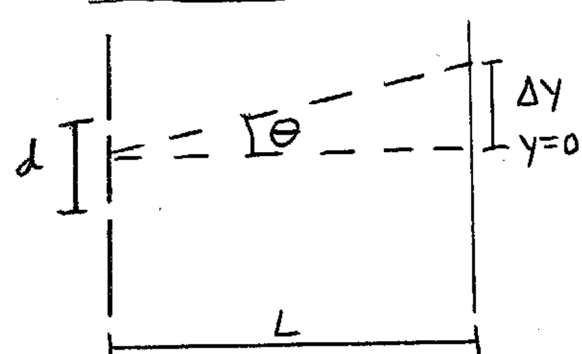
$$= \frac{\left(\frac{1}{2} + \frac{3}{4}\right)c}{1 + \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\right]} = \frac{\frac{5}{4}c}{1 + \frac{3}{8}} = \frac{\frac{5}{4}c}{\frac{11}{8}} = \frac{(5)(8)\,c}{(4)(11)}$$

$$= \frac{10}{11}c = \boxed{u_x = 0.909c \;\rightarrow \text{choice (d)}}$$

(6) Two slits separated by 0.10 mm are illuminated with green light ($\lambda = 540$ nm). Calculate the distance (in cm) from the brightest maximum in the center to the bright band with $m = 5$ if the screen is at a distance of 1.0 m from the two slits. (a) 2.3 (b) 2.5 (c) 2.7 (d) 2.1 (e) 2.0

Find – ΔY

Two-slit interference –



Find – Δy

$d \sin\theta = m\lambda$ for constructive interference, which makes maxima.

Since $\sin\theta \simeq \theta\,(\text{radians}) \simeq \frac{y}{L}$, for small θ,

$$\frac{dy}{L} \simeq m\lambda$$

$$y \simeq \frac{m\lambda L}{d}$$

$$\Delta y \simeq \frac{5\lambda L}{d} - \cancel{0\left(\frac{\lambda L}{d}\right)}$$

$=0$ for the central maximum, with $m=0$.

$$\Delta y \simeq \frac{5(5.40\times10^{-7}\,m)(1.0\,m)}{(0.1)(10^{-3}\,m)} = 0.027\,m = \boxed{\Delta y = 2.7\,cm \;\rightarrow \text{choice (c)}}$$

Problems: Box your final answers. No work = No credit on this part.

(A) (i) (4 points) A satellite is in a low-Earth orbit at an altitude $L = 200$ km $= 2.0 \times 10^5$ m above Earth's surface. Its camera has a circular aperture with a diameter $D = 2.5$ meters. This camera can detect visible light, which has a wavelength of 550 nm (where 1 nm $= 10^{-9}$ m). If the camera has adaptive optics, it can compensate for the blurring effects of Earth's atmosphere. However, diffraction imposes an absolute limit on the camera's image resolution, or in other words, the smallest detail the camera can see. How many meters in diameter (y) is the smallest object on Earth that this camera can resolve, or see clearly? [Hint: Since the satellite is many kilometers from Earth, $\sin\theta \approx y/L$.]

(ii) (4 points) This camera is a digital camera, much like those that are commercially popular today. Digital cameras don't use photographic film to detect light: they use electronic complementary metal oxide semiconductor (CMOS) detectors. CMOS detectors detect light by the photoelectric effect, in which photons that land on a silicon crystal knock electrons out of the crystal. These electrons are then caught and turned into images by digital electronics.

Silicon has a work function of 1.125 eV, where 1 eV = 1 electron volt $= 1.602 \times 10^{-19}$ J. What is the longest wavelength of light (in meters) that this CMOS detector can detect? [Hint: the cutoff frequency occurs where the kinetic energy of the electrons drops to zero.]

(iii) (4 points) Suppose the lens of the camera mentioned in the previous question has a coating of anti-reflection film on its front surface. This film is transparent to light, and has an index of refraction $n = 1.45$, which is less than than the index of refraction of the glass of which the lens is made, which is $n = 1.50$. This film's purpose is to prevent reflection from the front surface of the lens, so that the maximum possible intensity of light goes into lens, and therefore into the camera. What is the minimum thickness this film will need to have, for light with a wavelength $\lambda = 550$ nm?

(iv) (2 points) This satellite also has an infrared camera. It can detect thermal radiation (also called blackbody radiation, or Planck radiation), from objects that are not transparent, such as rocks, cars, or buildings. Suppose one of these objects radiates the maximum intensity of its thermal radiation at a wavelength $\lambda_{max} = 10$ microns, where 1 micron $= 1.0 \times 10^{-6}$ m. What is the temperature of this object, in Kelvins?

(v) (2 points) Suppose a thermal radiator of any Kelvin temperature T (not necessarily the radiator mentioned in the previous problem) radiates with a total intensity I. If this thermal radiator then doubles in Kelvin temperature, what will its total intensity I' now be, as a function of I? Assume the area of this thermal radiator is the same, at both Kelvin temperatures.

(vi) (4 points) During the *Apollo 11* mission in 1969, Buzz Aldrin set up a laser reflector on the Moon. Astronomers still bounce laser signals off of it. Precise timing of the laser signals can measure the distance to the Moon within 2 millimeters. Suppose a red laser on Earth emits a beam with a wavelength of 632.8 nm. This beam illuminates a spot on the Moon. The laser shines through a telescope on Earth that has a circular aperture with a diameter of 76.2 cm. What is the diameter of the spot the beam makes on the Moon? The Moon is 384,400 km from Earth. Ignore the effects of Earth's atmosphere.

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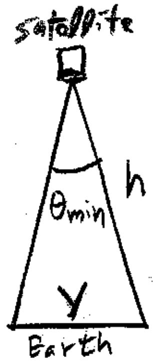
(i) Find − Y

This is diffraction through a circular aperture of $D = 2.5$ m, and the Rayleigh criterion. Use the small-angle approximation, $\sin\theta \simeq \theta$ (radians),



$$\sin\theta = \frac{Y}{h} \simeq \theta_{min} \simeq \frac{1.22\lambda}{D}$$

(the 1.22 is for a circular aperture),

$$Y \simeq \frac{1.22\lambda d}{h}$$

$$= \frac{(1.22)(5.5\times10^{-7}\,m)(2.0\times10^{5}\,m)}{(2.5\,m)}$$

$$\boxed{Y \simeq 0.054\,m = 5.4\,cm}$$

(ii) Find − λ_{cutoff}, for the photoelectric effect.

$$K_{max} = \frac{hc}{\lambda} - \phi$$

At cutoff, $K_{max} = 0$, so $\frac{hc}{\lambda} = \phi$.

$$\lambda = \frac{hc}{\phi} = \frac{1240\ eV\text{-}nm}{1.125\ eV}$$

6 $\boxed{\lambda_{cutoff} = 1100\ nm = 1.10\times10^{-6}\ m = 1.10\ microns}$

with 1 micron $= 10^{-6}$ m and 1 nm = 1 nanometer $= 10^{-9}$ m.

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(iii) Find – t, the thickness of the film.

This is thin-film interference: see page 973 of Serway & Jewett's textbook. For a non-reflective coating, we need constructive interference to let as much light as possible in, so:

$2t = (m + \frac{1}{2})\lambda$, $m = 0, 1, 2 \ldots$ (equation 36.11), and so:

$2t = \frac{\lambda}{2n}$ for $m = 0$, since we want the maximum film thickness, t.

$$\Rightarrow t \geq \frac{\lambda}{4n} = \frac{(5.50 \times 10^{-7} m)}{(4)(1.5)} = \boxed{\begin{array}{l} t = 9.48 \times 10^{-8} m \\ \quad = 94.8 \text{ nm}. \end{array}}$$

(iv) Find – T, for thermal radiation.

Wien's law for thermal radiation is: $\lambda_{max}(m) = \frac{2.898 \times 10^{-3} m \cdot K}{T(K)}$,

so $T(K) = 2.898 \times 10^{-3} m \cdot K / 1.0 \times 10^{-5} m = \boxed{T = 290 K}$

(v) For thermal radiators at any T, $I = \sigma T^4$,

where (sigma) $\sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$, the Stefan-Boltzmann constant.

$\frac{I'}{I} = \frac{\cancel{\sigma} T'^4}{\cancel{\sigma} T^4}$, so $I' = (T'/T)^4 I$ and with $T' = 2T$,

$\boxed{I' = 16 I}$, since $2^4 = 16$.

(vi) Find – Y, again for diffraction through a circular aperture, and the Rayleigh criterion, with $D = 0.762 m$.

$$\theta_{min} \simeq \frac{Y}{L} \simeq \frac{1.22\lambda}{D}$$

$$Y \simeq \frac{1.22\lambda L}{D} = \frac{(1.22)(6.328 \times 10^{-7} m)(3.844 \times 10^{8} m)}{0.762 m}$$

$\boxed{Y \simeq 389 m}$

(B) A quantum particle is a particle that has matter waves, such as an electron. A quantum particle with mass m is confined inside a one-dimensional box of width L. One wall of the box is at $x = 0$, and the other is at $x = L$. The particle is in the second excited state, with $n = 3$, and is described by the wave function $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right)$.

(i) (4 points) Suppose the particle makes a transition from $n = 3$ to the ground state, for which $n = 1$. To conserve energy (more precisely, mass-energy), the particle will emit a photon. What is the energy of this photon that is emitted? Write your answer as a function of h (Planck's constant), m, and L.

(ii) (4 points) What is the probability this particle is somewhere between $0 \le x \le \frac{L}{3}$? Use the integral $\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$. Give the value and the explicit analytical solution, or in other words, the worked integration for this problem.

(iii) (4 points) The wave function of this particle confined to a one-dimensional box of length L is $\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$. The walls of the box are at $x = 0$ and $x = L$. The particle can't be outside the box, so $\psi(x) = 0$ for $x \le 0$ and $\psi(x) = 0$ for $x \ge L$. Determine the constant A. [*Hint:* A useful integral is $\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$.]

(iv) (4 points) What is the probability for finding this particle somewhere between $0 \le x \le L$? [Hint: there is a relatively simple way to determine this.]

(v) (4 points) Two radio antennas are separated by a distance $d = 1$ km. Both antennas are $h = 50$ m tall, as shown in the figure below.

All the radio energy is radiated from the top of the antennas. Both antennas simultaneously transmit identical signals at the same wavelength. A car is traveling on the ground on a path that is a straight line between the two antennas. A radio in the car receives the signals. Neglect the effect of the curvature of Earth, since it is relatively small here (with the horizon curving downward by 7.85 cm per kilometer, or 8 inches per mile). When the car is 1/3 of the way between the antennas, it is at the position of the maximum with $m = 2$. What is the frequency of the signals?

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(i) For a QM particle in a box of width L,

$$E_n = \left(\frac{h^2}{8mL^2}\right) n^2, \quad n = 1, 2, 3 \ldots$$

$$E_{photon} = E_3 - E_1 = \left(\frac{h^2}{8mL^2}\right)(3^2 - 1^2)$$

$$= \frac{8h^2}{8mL^2} = \boxed{E_{photon} = \frac{h^2}{mL^2}}.$$

(ii)
$$P_{ab} = \int_{x=a}^{b} |\psi(x)|^2 dx = \int_{x=0}^{L/3} \frac{2}{L}\sin^2\left(\frac{3\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\frac{x}{2} - \frac{\sin(2ax)}{4a}\right]_{x=0}^{L/3} \quad \text{where } a = \frac{3\pi}{L} \text{ and } n = 3,$$

$$= \frac{2}{L}\left[\frac{x}{2} - \frac{L}{12\pi}\sin\left(\frac{6\pi x}{L}\right)\right]_{x=0}^{L/3}$$

$$= \frac{2}{L}\left[\frac{L}{6} - \frac{\overbrace{\sin(2\pi)}^{=0}}{12\pi} - 0 - 0\right] = \frac{2}{\cancel{L}}\left(\frac{\cancel{L}}{6}\right)$$

$\therefore \boxed{P = 1/3}$ that the particle is between $0 \le x \le L/3$.

(iii) Use normalization, since the particle must be somewhere between $-\infty < x < +\infty$,

So: $$P = 1 = \int_{x=-\infty}^{+\infty} |\psi(x)|^2 dx = A^2 \int_{-\infty}^{\infty} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= A^2 \int_{x=0}^{x=L} \sin^2(ax)\,dx \quad \text{with } a = \frac{n\pi}{L},$$

since $0 \le x \le L$,

so: $$1 = A^2\left[\frac{x}{2} - \frac{\sin(2ax)}{4a}\right]_{x=0}^{x=L}$$

$$= A^2\left[\frac{L}{2} - \underbrace{\frac{\sin(2\pi n\cancel{L}/\cancel{L})}{(4\pi n/L)}}_{=0} - 0 - 0\right]$$

$$= A^2\left[\frac{L}{2}\right] \Rightarrow \boxed{A = \sqrt{2/L}}.$$

(iv) The particle can't be outside the box, for which $x < 0$ and $x > L$,

8 so $\boxed{P = 1}$ that the particle is between $0 \le x \le L$.

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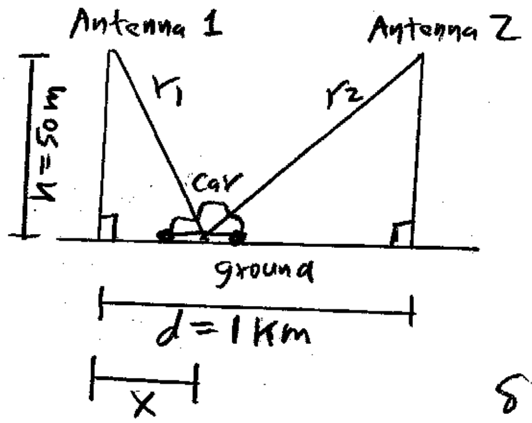
(v)



For constructive interference at the $m = 2$ maximum,

$$\delta = r_2 - r_1 \simeq d \sin\theta = m\lambda,$$

where $m = 2$.

$\delta = r_2 - r_1$, and by the Pythagorean theorem,

$$\delta = \sqrt{h^2 - (d-x)^2} - \sqrt{h^2 + x^2} = m\lambda.$$

$$\Rightarrow \quad 2\lambda = \sqrt{(50\,m)^2 + \left[\frac{2(1000\,m)}{3}\right]^2} - \sqrt{(50\,m)^2 + \left(\frac{1000\,m}{3}\right)^2}$$

$$\lambda = (331.47\,m)/2 = 165.7\,m$$

$$f = c/\lambda = (3.00 \times 10^8\,m/s)/(165.7\,m)$$

$$\boxed{f = 1.81 \times 10^6\,Hz = 1.81\,MHz}$$