1. The horizontal asymptote(s) of $f(x) = \dfrac{\sqrt{36x^4 + 5x - 2}}{3x^2 - 1}$ is/are

   (a) $\boxed{y = 2 \text{ only}}$

   (b) $y = 2$ and $y = -2$ only

   (c) $y = -2$ only

   (d) $f(x)$ has no horizontal asymptotes.

I accepted both (1a) and (1b) for this problem, though only (1a) is correct. The (modified) degrees of the numerator and denominator are both 2; therefore the limits at infinity will be determined (up to $\pm$) by the leading coefficients. We have

$$\lim_{x\to\infty} \frac{\sqrt{36x^4 + 5x - 2}}{3x^2 - 1} = \frac{6}{3} = 2$$

and

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4 + 5x - 2}}{3x^2 - 1} = +\frac{6}{3} = 2.$$

Note that regardless of whether we go to ∞ or $-\infty$, $\dfrac{\sqrt{36x^4 + 5x - 2}}{3x^2 - 1}$ will be positive!

To verify the above answers algebraically:

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\sqrt{36x^4 + 5x - 2}}{3x^2 - 1} &= \lim_{x\to\pm\infty} \frac{\sqrt{36x^4 + 5x - 2} \cdot \frac{1}{x^2}}{(3x^2 - 1) \cdot \frac{1}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{\sqrt{(36x^4 + 5x - 2)\frac{1}{x^4}}}{(3x^2 - 1) \cdot \frac{1}{x^2}}
\end{aligned}$$

(here is where we don't need to change the sign, even for $-\infty$, since $\sqrt{\frac{1}{x^4}}$ is equal to $\frac{1}{x^2}$ for <u>*all*</u> x)

$$\begin{aligned}
&= \lim_{x\to\pm\infty} \frac{\sqrt{36 + \frac{5}{x^3} - \frac{2}{x^4}}}{3 - \frac{1}{x^2}} \\
&= \frac{\sqrt{36}}{3} \\
&= 2.
\end{aligned}$$

2. If $g(t) = \dfrac{t^2}{\sin 3t}$, then $g'(t) =$

(a) $\dfrac{2t}{3\cos 3t}$

(b) $\dfrac{3t^2 \cos 3t - 2t \sin 3t}{\sin^2 3t}$

(c) $\boxed{\dfrac{2t \sin 3t - 3t^2 \cos 3t}{\sin^2 3t}}$

(d) $\dfrac{t^2 \sin 3t - 6t \cos 3t}{\sin^2 3t}$

Using the quotient rule, we get

$$\begin{aligned} g'(t) &= \frac{\sin 3t \cdot 2t - t^2 \cdot \cos 3t \cdot 3}{(\sin 3t)^2} \\ &= \frac{2t \sin 3t - 3t^2 \cos 3t}{\sin^2 3t}. \end{aligned}$$

3. If $f(-3) = 4$, $f'(-3) = 1$, $f'(2) = 5$, $g(-3) = 2$, and $g'(-3) = -1$, then $(f \circ g)'(-3) =$

(a) 4

(b) 2

(c) -2

(d) $\boxed{-5}$

The chain rule says $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. Therefore

$$(f \circ g)'(-3) = f'(g(-3)) \cdot g'(-3) = f'(2) \cdot g'(-3) = 5 \cdot -1 = -5.$$

4. The function $f(x) = 2x^3 + x - 1$

   (a) has exactly 3 real roots

   (b) $\boxed{\text{has exactly 1 real root, which is between 0 and 1}}$

   (c) has exactly 1 real root, which is between -1 and 0

   (d) has no real roots

   Since

$$\begin{aligned} f(0) &= -1 \leq 0 \\ f(1) &= 2 \geq 0, \end{aligned}$$

   we know there is at least one real root of $f(x)$ between 0 and 1, by the Intermediate Value Theorem.

   To see that there is only one real root, we use Rolle's Theorem: if $f(a) = f(b) = 0$, then there is a number c between a and b such that $f'(c) = 0$, since $f(x)$ is continuous and differentiable everywhere. But $f'(x) = 6x^2 + 1$, which is not equal to 0 for any value of x. This contradicts our hypothesis; therefore there is only one real root.

5. If $\sin y = \dfrac{3x^2}{x+1}$, then $\dfrac{dy}{dx} =$

   (a) $\boxed{\dfrac{3x^2 + 6x}{(x+1)^2 \cos y}}$

   (b) $\dfrac{6x}{\cos y}$

   (c) $\cos^{-1}\left(\dfrac{3x^2 + 6x}{(x+1)^2}\right)$

   (d) $\cos^{-1}\left(\dfrac{3x^2}{x+1}\right)\dfrac{3x^2 + 6x}{(x+1)^2}$

   Using implicit differentiation, we get

$$\begin{aligned} \cos y \frac{dy}{dx} &= \frac{(x+1)6x - 3x^2(1)}{(x+1)^2} \\ &= \frac{3x^2 + 6x}{(x+1)^2}; \end{aligned}$$

   therefore $\dfrac{dy}{dx} = \dfrac{3x^2 + 6x}{(x+1)^2 \cos y}$.

6. The linearization of the function $f(x) = x^3$ at $a = 2$ is

   (a) $L(x) = 3x^2$

   (b) $\boxed{L(x) = 12x - 16}$

   (c) $L(x) = 6x - 8$

   (d) $L(x) = 3x - 2$

   The linearization of a function $f(x)$ at $x = a$ is simply the equation of the tangent line to the function at a. The formula is

   $$L(x) = f'(a)(x - a) + f(a).$$

   We know that $f'(x) = 3x^2$, so $f'(2) = 12$. Also $f(2) = 8$. Therefore

   $$\begin{aligned} L(x) &= 12(x - 2) + 8 \\ L(x) &= 12x - 16. \end{aligned}$$

7. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. Then the volume at this instant is decreasing at a rate of

   (a) 80 cm/min

   (b) $\boxed{80 \text{ cm}^3/\text{min}}$

   (c) 40 cm^3/min

   (d) 40 kPa/min

   Differentiating $PV = C$ with respect to t, we get

   $$P\frac{dV}{dt} + V\frac{dP}{dt} = 0,$$

   using the product rule on the left-hand side and the fact that C is a constant on the right. Then we plug in

   $$\begin{aligned} V &= 600 \\ P &= 150 \\ \frac{dP}{dt} &= 20 \end{aligned}$$

   to get

   $$150 \cdot \frac{dV}{dt} + 600 \cdot 20 = 0.$$

   Solving for $\frac{dV}{dt}$, we get $\frac{dV}{dt} = -80$, which makes sense since the volume is decreasing. The change in volume per unit time is measured in cm^3/min, so the volume is decreasing at a rate of 80 cm^3/min.

8. If $x_1 = 0$ is a first approximation of a root of $f(x) = x^5 + 2x + 1$, then using Newton's Method the second approximation is $x_2 =$

   (a) $-\frac{2}{5}$

   (b) $\frac{2}{5}$

   (c) $\boxed{-\frac{1}{2}}$

   (d) $\frac{1}{2}$

   Note that $f'(x) = 5x^4 + 2$, so $f'(0) = 2$. Also $f(0) = 1$. Using the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

   we get $x_2 = 0 - \dfrac{f(0)}{f'(0)} = -\dfrac{1}{2}$.

9. From the graph of $f(x)$ shown, $f(x)$

   (a) is an even function

   (b) $\boxed{\text{is an odd function}}$

   (c) is neither an even nor an odd function

   (d) may or may not be even or odd; it is impossible to tell without the formula.

   See me for the picture for this problem. The picture shows a graph which is symmetric about the origin. Therefore it is the graph of an odd function.

10. Tinkle Winkle Company makes wooden music boxes with glass tops. Wood costs \$4 per square foot and glass costs 2.50 per square foot. The music mechanism requires 10 cubic inches of space inside each music box. Tinkle Winkle Company wishes to figure out the dimensions of a music box which will minimize the cost per box. The objective of the problem is

   (a) to sell music boxes cheaply

   (b) to make sure each music box is big enough for the music mechanism

   (c) $\boxed{\text{to minimize the cost of producing the music boxes}}$

   (d) to maximize the volume of each music box

   The *objective* of a max-min problem is a statement about what we are trying to maximize or minimize. The company wants to minimize the cost of the music boxes. Their goal is not to maximize the volume of the boxes; they only require that the boxes contain 10 in.^{3} of space. While it is true that the company wants to sell music boxes cheaply and to make sure each music box is big enough for the music mechanism, these are not correctly stated *objectives* for the problem.

BONUS. (5 points) The 100th derivative of $f(x) = x \sin x$ is

(a) $x \cos x + 100 \sin x$

(b) $x \cos x - 100 \sin x$

(c) $x \sin x + 100 \cos x$

(d) $\boxed{x \sin x - 100 \cos x}$

(e) None of these.

We have

$$\begin{aligned} f'(x) &= x \cos x + \sin x \\ f''(x) &= -x \sin x + \cos x + \cos x = -x \sin x + 2 \cos x \\ f'''(x) &= -x \cos x - \sin x - \sin x - \sin x = -x \cos x - 3 \sin x \\ f^{(4)}(x) &= x \sin x - \cos x - \cos x - \cos x - \cos x = x \sin x - 4 \cos x \\ &\cdots \end{aligned}$$

Continuing the pattern, we will get

$$f^{(100)}(x) = x \sin x - 100 \cos x.$$