Math 75 - Sample Questions for Midterm III - Solutions

1. $\int_{-2}^{2} \sqrt{4-x^2}\,dx =$

   (a) $-\frac{1}{6}$
   (b) 0
   (c) $\boxed{2\pi}$
   (d) does not exist.

   The curve $y = \sqrt{4-x^2}$ is the upper half of a circle with radius 2 centered at $(0,0)$. Therefore the integral $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ represents the area of a semicircle of radius 2, which is $\frac{1}{2} \cdot \pi \cdot 2^2 = 2\pi$.

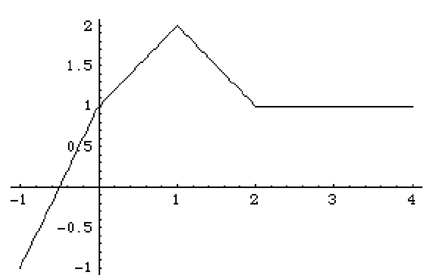
2. $\int_{0}^{\pi/4} \sec x \tan x\,dx =$

   (a) $\boxed{\sqrt{2}-1}$
   (b) $\sqrt{2}$
   (c) $1-\frac{\sqrt{2}}{2}$
   (d) does not exist.

   The function $f(x) = \sec x \tan x$ is defined on the interval $\left[0, \frac{\pi}{4}\right]$, so the integral exists. Using the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_{0}^{\pi/4} \sec x \tan x\,dx &= \sec x\Big|_{0}^{\pi/4}\\ &= \sec\left(\frac{\pi}{4}\right) - \sec(0)\\ &= \sqrt{2}-1.\end{aligned}$$

For #3-4, use the graph of $f(x)$ shown below to answer the questions.



3. $\displaystyle\int_{-1}^{0} f(x)\,dx \;=$

   (a) -1
   (b) $\boxed{0}$
   (c) 1
   (d) 2

Between $x=-1$ and $x=0$ there is just as much area (between $f(x)$ and the x-axis) above the x-axis as below. Therefore the net area is 0.

4. $\displaystyle\int_{2}^{1} f(x)\,dx \;=$

   (a) $\boxed{-\frac{3}{2}}$
   (b) $\frac{3}{2}$
   (c) -1
   (d) 1

The area under $f(x)$ from $x=1$ to $x=2$ is $\frac{3}{2}$. But the limits of integration have been reversed. So

$$\begin{aligned}\int_2^1 f(x)\,dx &= -\int_1^2 f(x)\,dx\\ &= -\frac{3}{2}.\end{aligned}$$

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5. $\displaystyle\int_0^{\pi} \cos\left(5\theta - \frac{\pi}{2}\right)\,d\theta =$

   (a) $\boxed{\dfrac{2}{5}}$
   (b) $-\dfrac{2}{5}$
   (c) $\dfrac{\pi}{2}$
   (d) $-\dfrac{\pi}{2}$

Let $u = 5\theta - \frac{\pi}{2}$. Then $du = 5\ d\theta$. We also have that when $\theta = 0$, $u = -\frac{\pi}{2}$ and when $\theta = \pi$, $u = 5\pi - \frac{\pi}{2} = \frac{9\pi}{2}$. So we get

$$\begin{aligned}
\int_0^\pi \cos\left(5\theta - \frac{\pi}{2}\right)\ d\theta &= \frac{1}{5}\int_0^\pi 5\cos\left(5\theta - \frac{\pi}{2}\right)\ d\theta \\
&= \frac{1}{5}\int_{-\pi/2}^{9\pi/2} \cos u\ du \\
&= \frac{1}{5}\sin u\Big|_{-\pi/2}^{9\pi/2} \\
&= \frac{1}{5}\left(\sin\left(\frac{9\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{1}{5}\left(1 - (-1)\right) = \frac{2}{5}.
\end{aligned}$$

6. $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\ dx =$

   (a) $x^{4/3} - \tan^2 x + C$

   (b) $\boxed{\frac{3}{4}x^{4/3} - \tan x + C}$

   (c) $\frac{3}{4}x^{4/3} - \frac{1}{\sin^2 x} + C$

   (d) $\frac{1}{3}x^{-2/3} - 2\sec^2 x \tan x + C$

$\sqrt[3]{x} = x^{1/3}$, so $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\ dx = \frac{3}{4}x^{4/3} - \tan x + C$ (this is straight out of the formulas in §5.4, p. 347).

7. $\displaystyle\int x^2(5 - x^3)^{20}\ dx =$

   (a) $\frac{1}{63}(5 - x^3)^{21} + C$

   (b) $\boxed{-\frac{1}{63}(5 - x^3)^{21} + C}$

   (c) $-\frac{20}{3}(5 - x^3)^{19} + C$

   (d) $\frac{x^2}{21}(5 - x^3)^{21} + C$

Let $u = 5 - x^3$. Then $du = -3x^2\ dx$, and we have

$$\begin{aligned}
\int x^2(5 - x^3)^{20}\ dx &= -\frac{1}{3}\int -3x^2(5 - x^3)^{20}\ dx \\
&= -\frac{1}{3}\int u^{20}\ du \\
&= -\frac{1}{3}\cdot\frac{1}{21}u^{21} + C \\
&= -\frac{1}{63}(5 - x^3)^{21} + C.
\end{aligned}$$

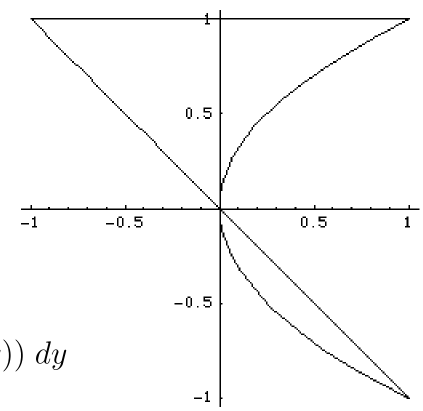
8. What expression best represents the area between $x = y^2$ and $x = -y$ from $y = -1$ to $y = 1$?

(a) $\displaystyle\int_{-1}^{0} (y^2 + y)\, dy \; + \int_{0}^{1} (-y - y^2)\, dy$

(b) $\boxed{\displaystyle\int_{-1}^{0} (-y - y^2)\, dy \; + \int_{0}^{1} (y^2 + y)\, dy}$

(c) $\displaystyle\int_{-1}^{1} (y^2 + y)\, dy$

(d) $\displaystyle\int_{-1}^{0} (y^2 - y)\, dy \; + \int_{0}^{1} (y - y^2)\, dy$



The region described is in two pieces, as shown.

The two curves cross at $y = 0$.

From $y = -1$ to $y = 0$, $x = -y$ is on the right.

From $y = 0$ to $y = 1$, $x = y^2$ is on the right.

Therefore the area is

$$\int_{-1}^{0} (-y - y^2)\, dy \; + \int_{0}^{1} (y^2 - (-y))\, dy$$
$$= \int_{-1}^{0} (-y - y^2)\, dy \; + \int_{0}^{1} (y^2 + y)\, dy.$$

9. The area enclosed by the curves $y = \frac{1}{x^3}$, $y = \frac{1}{x^2}$, and $x = 2$ is

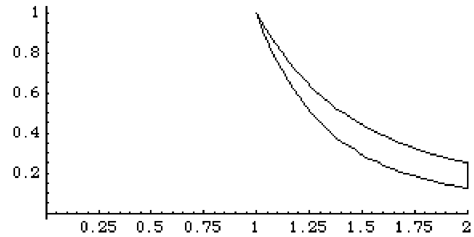
(a) $\displaystyle\int_{0}^{2} \left(\frac{1}{x^3} - \frac{1}{x^2}\right)\, dx$

(b) $\displaystyle\int_{0}^{2} \left(\frac{1}{x^2} - \frac{1}{x^3}\right)\, dx$

(c) $\displaystyle\int_{1}^{2} \left(\frac{1}{x^3} - \frac{1}{x^2}\right)\, dx$

(d) $\boxed{\displaystyle\int_{1}^{2} \left(\frac{1}{x^2} - \frac{1}{x^3}\right)\, dx}$

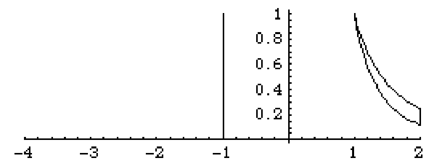
The region described is shown.



Notice that the curve $\frac{1}{x^2}$ is on top between $x = 1$ and $x = 2$. Therefore the area is

$$\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx.$$

10. The volume of the solid formed by rotating the region enclosed by the curves $y = \frac{1}{x^3}$, $y = \frac{1}{x^2}$, and $x = 2$ about the line $x = -1$ is



(a) $2\pi \displaystyle\int_0^2 (x+1) \left(\frac{1}{x^3} - \frac{1}{x^2} \right) \, dx$

(b) $\boxed{2\pi \displaystyle\int_1^2 (x+1) \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx}$

(c) $2\pi \displaystyle\int_0^2 (x-1) \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx$

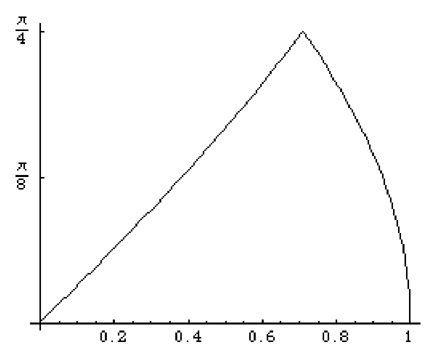
(d) $2\pi \displaystyle\int_1^2 (1-x) \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx$

The region being rotated is shown above with the axis of rotation. It is the same region as in #9. Since the region is formed from functions of x and is being rotated about a vertical axis, we use the **shell method**:

At any x between 1 and 2, the height of the shell is $h = \frac{1}{x^2} - \frac{1}{x^3}$ and the radius is $r = x+1$. Therefore the volume is

$$2\pi \int_1^2 (x+1) \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx.$$

11. The volume of the solid formed by rotating the region shown about the y-axis is



(a) $2\pi \displaystyle\int_0^{\pi/4} y \left(\sin y - \cos y \right) \, dy$

(b) $\pi \displaystyle\int_0^{\pi/4} \left(\cos y - \sin y \right)^2 \, dy$

(c) $\boxed{\pi \displaystyle\int_0^{\pi/4} \left(\cos^2 y - \sin^2 y \right) \, dy}$

(d) $2\pi \displaystyle\int_0^{\pi/4} y \left(\cos y - \sin y \right) \, dy$

We can see from the answer choices that the two curves shown are $x = \cos y$ and $x = \sin y$. From our knowledge of these curves we know that $x = \cos y$ is the curve on the right (it passes through the point $(1, 0)$) and $x = \sin y$ is the curve on the left (it passes through the point $(0, 0)$).

Since the region is formed by functions of y and is being rotated about a vertical axis, we use the **disk method**:

At any y between 0 and $\frac{\pi}{4}$, the outer radius of the disk is $R = \cos y$ and the inner radius of the disk is $r = \sin y$. Therefore the volume is

$$\begin{aligned} &\pi \int_0^{\pi/4} \left((\cos y)^2 - (\sin y)^2\right)\, dy \\ &= \pi \int_0^{\pi/4} \left(\cos^2 y - \sin^2 y\right)\, dy. \end{aligned}$$

12. Lois Lane, whose mass is 50 kg, is hanging from a 20-meter rope tied to a crane. Superman is at the top of the crane. In order to rescue Lois, he must pull the rope all the way up to the top of the crane. If the rope has a mass of 10 kg, then the work Superman must do in order to rescue Lois is

    (a) 10,780 N

    (b) $\boxed{\text{10,780 J}}$

    (c) 9,800 N

    (d) 9,800 J

The sneaky way to determine the answer is to notice that

- The work done (metric system) is measured in Joules (J), so the answer is either (b) or (d).
- Lois Lane's weight is $50 \cdot 9.8 = 490$ N, so the work required to lift only her is $490 \cdot 20 = 9800$ N, since the rope is 20 m long. So the answer must be (b) since Superman also has to pull the rope up!

But here's how to do the integral:

The rope weighs $10 \cdot 9.8 = 98$ N, or $\frac{98}{20} = 4.9$ Newtons per meter. So if Superman has pulled up x meters of rope, the weight of the rope he has pulled up is $4.9x$. Therefore the weight he is still pulling is $98 - 4.9x = 4.9(20 - x)$ Newtons, in addition to Lois's 490 N.

The total work done, then, is

$$\begin{aligned}
W &= \int_0^{20} (4.9(20-x)+490)\ dx \\
&= 4.9\int_0^{20} (20-x+100)\ dx \\
&= 4.9\int_0^{20} (120-x)\ dx \\
&= 4.9\left(120x-\frac{1}{2}x^2\right)\Big|_0^{20} \\
&= 4.9\left(120\cdot 20-\frac{1\cdot 20^2}{2}\right)-(0-0) \\
&= 4.9(2400-200) \\
&= 4.9(2200) = 10,780\text{ J.}
\end{aligned}$$

13. The temperature (in °F) t hours after 12 noon is $f(t) = 50 + 14\sin(\frac{\pi t}{2})$. The average temperature from 1 pm to 10 pm is

(a) $\frac{1}{9}(500 + \frac{28}{\pi})$

(b) $\boxed{\frac{1}{9}(450 + \frac{28}{\pi})}$

(c) $500 + \frac{28}{\pi}$

(d) $450 + \frac{28}{\pi}$

The average temperature is the average value of the function $f(t)$ from $t = 1$ to $t = 10$, which is

$$\begin{aligned}
\frac{1}{10-1}\int_1^{10} 50+14\sin\left(\frac{\pi t}{2}\right)\ dt &= \frac{1}{9}\left(50t-\frac{14\cdot 2}{\pi}\cos\left(\frac{\pi t}{2}\right)\right)\Big|_1^{10} \\
&= \frac{1}{9}\left((50\cdot 10-\tfrac{28}{\pi}\cos(5\pi))-(50-\tfrac{28}{\pi}\cos(\tfrac{\pi}{2}))\right) \\
&= \frac{1}{9}\left(450-\tfrac{28}{\pi}(-1)+\tfrac{28}{\pi}(0)\right) \\
&= \frac{1}{9}\left(450+\frac{28}{\pi}\right).
\end{aligned}$$