Section 5.3 - The Fundamental Theorem of Calculus - p. 337 Stewart, 4th Ed.

Read p. 337 paragraph.

Fundamental Theorem of Calculus (part 1): The area under $f(x)$ from a to x is an antiderivative of $f(x)$. (we'll come back to this)

Fundamental Theorem of Calculus (part 2): If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$.

Example. $\int_1^4 x^2 dx$.

Let $f(x) = x^2$. An antiderivative is $F(x) = \frac{1}{3}x^3$.

$F(4) =$ _____, $F(1) =$ _____. So $\int_1^4 x^2 dx = F(4) - F(1) =$ _____ $-$ _____ $=$ _____.

Example. $\int_{-1}^1 (x^3 - x^2 - 1)\ dx$.

$$\int_{-1}^1 (x^3 - x^2 - 1)\ dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x \Big|_{-1}^1$$

(note the notation — one also sees $\Big]_{-1}^1$ or $\Big[\quad\Big]_{-1}^1$)

$$= \underline{\hspace{8cm}}.$$

Example. $\int_0^\pi 2\cos t\ dt$.

Solution:

Example. $\int_{-1}^1 \frac{1}{x^2}\ dx = ?$

Careful! The answer is not $\left[-\frac{1}{x}\right]_{-1}^1$!

In fact, $\int_{-1}^1 \frac{1}{x^2}\ dx$ "does not exist"! (Technically it's called "improper," but for Math 75 we'll say D.N.E.)

Notes:

Fact. Using F.T.C. (Fundamental Theorem of Calculus), we get $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$.

This fact is especially good when dealing with **F.T.C, part 1**:

F.T.C., part 1. If $f(x)$ is continuous on $[a, b]$, then $f(x)$ has an *antiderivative function* which is the area under $f(x)$ from a to x, as long as x represents a number between a and b.

Example. If $F(x) = \int_{45}^{x} 3t^2\, dt$, then $F'(x) = 3x^2$.

Check: $F(x) = t^3\big|_{45}^{x} = x^3 - (45)^3 + C$, so $F'(x) = 3x^2$ since C is a constant.

Notice that the '45' did not affect the answer!

Example. If $F(x) = \int_{a}^{x} \sqrt{1+2t}\, dt$, then $F'(x) =$ ____________, *no matter what a is.*

Nice to know, since we do not yet know how to evaluate the antiderivative of $\sqrt{1+2t}$!

Example. If $F(x) = \int_{x}^{b} \cos t\, dt$, what is $F'(x)$?

Careful! The answer is not $\cos x$! It is $-\cos x$!

Reason:

Example. If $F(x) = \int_{a}^{x^2} 3t^2\, dt$, what is $F'(x)$?

Ack! What to do with the x^2??
Since we're actually taking a derivative, the **chain rule** comes into play here.

Recall the **chain rule**: $[G(u(x))]' = G'(u(x)) \cdot u'(x)$. We will use this as follows: let $u = x^2$. Then $u' = 2x$. Also let

$$G(u) = \int_{a}^{u} 3t^2\, dt,$$

so $G'(u) = 3u^2$ by F.T.C. (part 1). Notice that $F(x) = G(u(x))$. Thus by the chain rule we have

$$\begin{aligned} F'(x) &= \big[G(u(x))\big]' = G'(u(x)) \cdot u'(x) \\ &= 3u^2 \cdot 2x = 3x^4 \cdot 2x = 6x^5. \end{aligned}$$

Moral: whenever there is something other than x as one of the limits of integration, "treat it like an x" in the chain rule sense (replace all x's in the first part of your answer by whatever it is), and then tack on the derivative of whatever of it is.

"Quick and dirty" method for the above example:

1. The x^2 is in the *upper* limit, so we don't have to worry about negatives.

2. $F'(x) = 3(x^2)^2 \cdot (x^2)' = 3x^4 \cdot 2x = 6x^5$.