Section 5.5 - The Substitution Rule - p. 356 Stewart, 4th Ed.

The **substitution rule** is the opposite of the chain rule. Consequently, it will be your most important integration tool.

Recall. $\frac{d}{dx}(x^2-5)^3 = 3(x^2-5)^2(2x)$. This is because the chain rule states that

$$[f(g(x))]' = f'(g(x))g'(x).$$

To go backwards, we will have to look for something like $f'(g(x))g'(x)$ in the integrand. That's the **substitution rule**.

Example. $\displaystyle\int 3(x^2-5)^2(2x)\,dx = (x^2-5)^3 + C$, of course.

Study the following example on your own to see how it works. In class I'll show how we *really* do these problems.

Example. $\displaystyle\int 2x\sqrt{1+x^2}\,dx.$

Can we find $f'(g(x))$ and $g'(x)$ in there somewhere?

Idea: $g(x) = 1+x^2$. Then $g'(x) = 2x$, and we have $\displaystyle\int g'(x)\sqrt{g(x)}\,dx = \int g'(x)f'(g(x))\,dx$, where $f'(x) = \sqrt{x}$. If $f'(x) = \sqrt{x}$, then $f(x) = \frac{2}{3}x^{3/2} + C$, so the antiderivative is

$$f(g(x)) + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

Check: $\frac{d}{dx}(\frac{2}{3}(1+x^2)^{3/2} + C) = (1+x^2)^{1/2}(2x)$. Huzzah!

Here's how we really do these: u**-substitution**.

Example. $\displaystyle\int 2x\sqrt{1+x^2}\,dx.$

Let $u = 1+x^2$. Then $\frac{du}{dx} = 2x$ (here $u = g(x)$ in the chain rule).
Recall. "Cheating with differentials": multiply both sides by dx. We get

$$du = 2x\,dx.$$

Then what we have for the integral is $\displaystyle\int \sqrt{u}\,du$, which looks a *lot* easier to solve!

Procedure: find an antiderivative in u, then "back-substitute" to get back to x's.

$$\begin{aligned}\int \sqrt{u}\,du &= \int u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2} + C\end{aligned}$$

(Here's the **back-substitution** part: we know $u = x^2 + 1$.)

$$= \frac{2}{3}(x^2 + 1)^{3/2} + C.$$

Again you see why Leibniz's notation is useful!

So the **substitution rule** says, "u-substitution works." Or, "We can cheat with differentials" because it amounts to reversing the chain rule.

Example. $\displaystyle\int 3x^2(x^3 + 1)^4\, dx.$

Let $u =$ ______________. Then $du =$ ______________, and we are home free.

Fill in your complete solution:

What to substitute? This is always the toughest question to answer. You should try for

- something that will make the problem simpler
- something whose derivative (up to a constant multiple) appears in the integrand.

Example. $\int \sin(2x + 1)\, dx.$

We really want $u = 2x + 1$. But then $du = 2\, dx$. How are we going to get the extra "2"?

Answer: just put it in, then multiply by $\frac{1}{2}$ to compensate (I call this "futzing with the constant").

$\int \sin(2x + 1)\, dx =$

If "futzing with the constant" is too strange, you can also try the "solving for what you want" method:

Example. $\displaystyle\int t(4t^2 - 5)^3\, dt$

Let $u = 4t^2 - 5$. Then
$du = 8t\, dt$.
We want $t\, dt$. So divide both sides by 8: $\frac{du}{8} = t\, dt$. Then put it in:

$\int t(4t^2 - 5)^3\, dt =$

Use whichever method works best for you.

Definite integrals with u-substitution: **be careful!**
Example. What is wrong with the following calculation:

$$\begin{aligned}\int_0^2 t^2 \cos(t^3)\, dt &= \frac{1}{3}\int_0^2 3t^2 \cos(t^3)\, dt \\ &= \frac{1}{3}\int_0^2 \cos u\, du \\ &= \frac{1}{3}\sin u \Big|_0^2 \\ &= \frac{1}{3}(\sin 2 - \sin 0) = \frac{\sin 2}{3}.\end{aligned}$$

Answer: we plugged x-limits in for u ! Remember, we must either

- back-subsitute or
- change x-limits into u-limits

before we can use the F.T.C.

What does "change x-limits into u-limits" mean?
It means figuring out what u is when x is the upper and lower limits.

In the above example, $u = t^3$. What is u when x is 0? When x is 2?
Lower limit: $u = 0^3 = 0$.
Upper limit: $u = 2^3 = 8$.
So the correct answer is