Section 6.1 - Areas Between Curves (lecture 1) - p. 433 Stewart, 4th Ed.
Instructor Edition

Chapter 6 is called, "Applications of Integration." Now that we have some decent tools to evaluate integrals, we are going to explore some of the useful things we can calculate using them. The first important application is **volumes** of solids; the other main one is **work** (in the physics sense).

In §6.1 we study how to find the area *between* two curves, which, as it turns out, will be crucial in order to calculate volumes of solids.

Recall. The area under $f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx$. When we say "the area under $f(x)$" we really mean "the area between $f(x)$ and the x-axis." How would we find the area between two arbitrary (continuous) functions?

Example. What is the area between $f(x)$ and $g(x)$, as shown ($f(x)$ is the one on the top), between $x = 1$ and $x = 2.5$?

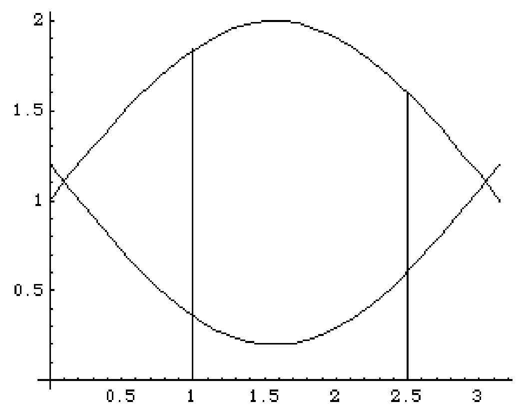

Figure 1: Area between two curves

Answer. Area under $f(x)$ − Area under $g(x)$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b (f(x) - g(x))\,dx.$$

Notice that the "curve on top", $f(x)$ in this case, came first.

Example. Find the area between $f(x) = x^3$ and $g(x) = x^2$ from 0 to 1.

Solution. Draw graphs of $f(x) = x^3$ and $g(x) = x^2$ from 0 to 1:

The graph of ____________ is on top between 0 and 1.
Therefore the area is

What if the function which is on top changes from a to b?

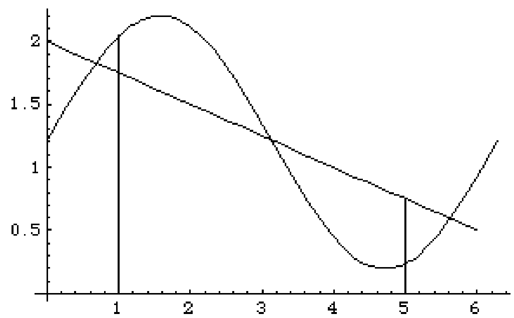

Figure 2: $f(x)$ and $g(x)$ "change places"

Answer. Do it in pieces: Area $= \int_1^c g(x) - f(x)\,dx + \int_c^5 f(x) - g(x)\,dx$. You have to figure out what the value of c is!

Example. Find the area between $y = x^3$ and $y = \frac{1}{2}(x+1)$ from 0 to 2.

Solution. Draw graphs of $y = x^3$ and $y = \frac{1}{2}(x+1)$ from 0 to 2:

The graph of ____________ is on top at 0. But ____________ is on top at 2. So they must have crossed!

Find the point(s) of intersection: set $\frac{1}{2}(x+1) = x^3$.

$x^3 - \frac{1}{2}x - \frac{1}{2} = 0.$

Remember the Intermediate Value Theorem? Use it to locate a root, or use *Mathematica* to graph both functions to see where the two functions intersect, then try a few values.

From the picture (Figure 3), it looks close to $x = 1$. Sure enough, $1^3 - \frac{1}{2}(1) - \frac{1}{2} = 0$. So the area is

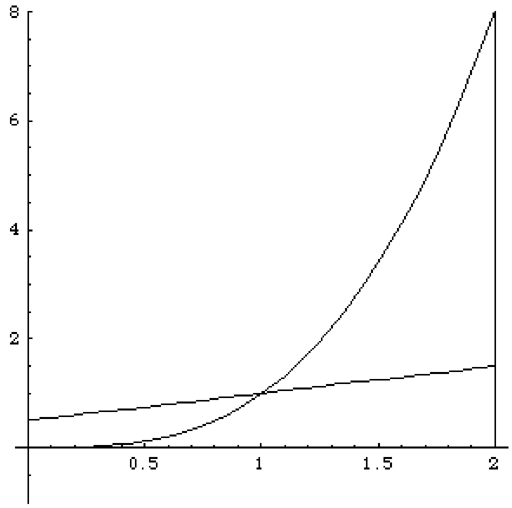

Figure 3: $\frac{1}{2}(x+1)$ and x^3 intersect at approximately $x = 1$

Other language for finding area between curves: "Find the area *enclosed by* the curves ...".

Tactic. Draw a graph to see which region they're actually talking about (where the curves are likely to intersect, which function is on top and when, etc.), compute the exact intersection points, if possible, and compute the integral.

Example. Find the area enclosed by $y = x^2$ and $y = x^3$.

Recall that x^2 and x^3 intersect at 0 and 1. Do they intersect anywhere else?

Setting $x^2 = x^3$, we see that $x^2(x-1) = 0$. Therefore the only solutions are $x = 0$ and $x = 1$.

So the area we are being asked to find is the same one we found previously.