Section 6.1 - Areas Between Curves (lecture 1) - p. 371 Stewart, 4th Ed.

Recall: The area between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is $\displaystyle\int_a^b (f(x) - g(x))\ dx$ when $f(x)$ is "above" $g(x)$ and is $\int_a^b (g(x) - f(x))\ dx$ when $f(x)$ is "below" $g(x)$. In other words,

$$(\text{ Area between } f(x) \text{ and } g(x) \text{ from } x = a \text{ to } x = b\ ) = \int_a^b |f(x) - g(x)|\ dx.$$

Here we are integrating **with respect to** x; in other words,

- The limits of integration are x-values
- The integrand is a function of x.

We write dx at the end of the integral to indicate this scenario.

Sometimes it's easier to **integrate with respect to** y. This is usually the case when the curves in question are not functions of x. I like to call this "sideways integration."

Example: Find the area enclosed by the curves $x = y^2$ and $x = 2y$.

The curves look like $y = x^2$ and $y = 2x$, except sideways:

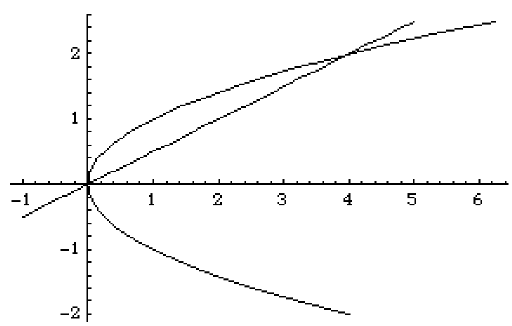

Figure 1: Area enclosed by $x = y^2$ and $x = 2y$

In "sideways integration," instead of looking for the "curve on top," we look for the "curve on the right." This is because we need to figure out which curve has larger x-values, not y-values.

Notice that from $y = 0$ to $y = 2$, the "curve on the right" is $x = 2y$. So the area is

Example. Find the area enclosed by the curves $y = 2x$, $x = \cos y$, $y = 0$, and $y = \frac{\pi}{4}$.

What does the curve $x = \cos y$ look like? Is y a function of x here? _____ , because

_______________________________________.

Idea: Write the curves as functions of y:

$$x = \frac{y}{2}$$
$$x = \cos y$$

It would be difficult to figure out exactly where these curves intersected, which curve was on the right and when, etc. But we could write a formula for the area between the curves from $y = 0$ to $y = \frac{\pi}{4}$:

$$\int_0^{\pi/4} \left| \frac{y}{2} - \cos y \right| dy.$$

Encoded in this formula is the notion that *whatever curve is on the right at a given y-value, that's the one that will be written first when we evaluate the integral.*

The curves actually look like (Figure 2).

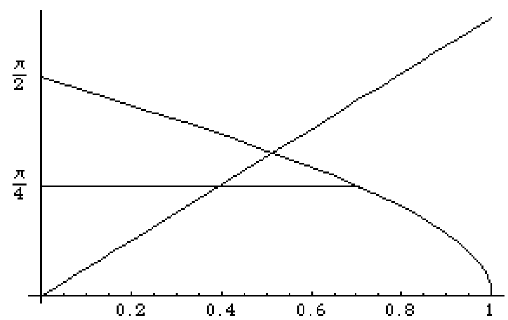

Figure 2: Area enclosed by $y = 2x$, $x = \cos y$, $y = 0$, and $y = \frac{\pi}{4}$

Notice that from $y = 0$ to $y = \frac{\pi}{4}$, the "curve on the right" is $x = \cos y$. So the area is

Summary:

- Figure out whether to integrate with respect to x or with respect to y.
- If y, rewrite equations of curves as functions of y (solve for x).

- Figure out which function is more positive (in either the x or the y sense) and where. It can be helpful to draw a graph. Find intersection points and split up the integral, if necessary.
- Evaluate the integral and do a reality check.

Example. Find the area enclosed by $y = -x + 1$ and $y^2 = 3x - 1$.

Question. Should we integrate with respect to x or y?
Answer. _____ is better, since ____.

Rewrite. The curves are $x =$ _____________and $x =$ _____________.
Draw a graph:

Notice that $x = 1 - y$ is further to the right. Next, locate the points of intersection: set

$$1 - y = \frac{1}{3}y^2 + \frac{1}{3}$$

The intersection points are approximately at _____________ and _____________.

Therefore the area is approximately

(This integral is not hard, but it's easy to make mistakes. Try it yourself and check the answer.)