Section 6.2 - Volume - p. Stewart, 4th Ed. (lecture 1)

We will ultimately learn how to define the volume of any solid by an integral, but we will chiefly focus on the volume of a **solid of rotation**, *i.e.* a solid formed by rotating a region about an axis.

Example. A solid sphere is formed when the region bounded by $y = \sqrt{1 - x^2}$ and the x-axis is rotated about the x-axis:

Question. What is formed when the rectangle shown in Figure 1 is rotated about the y-axis? About the line $x = -2$?

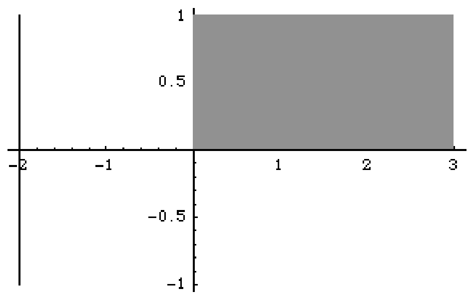

Figure 1: A rectangle to be rotated about an axis

Answer:

- About the y-axis: ______________________________.
- About $x = -1$: ______________________________.

We know the formulas for these volumes. But how would we find the volume, say, of the solid formed by rotating the region enclosed by $f(x) = \sqrt{x}$, $y = 0$, and $x = 4$ about the x-axis?

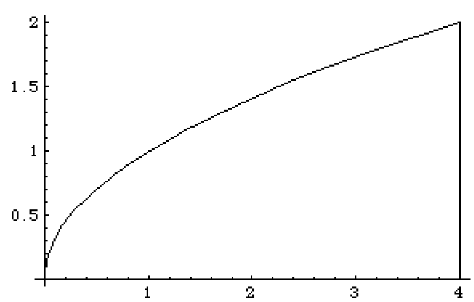

Figure 2: A more complicated region to be rotated about an axis

To answer this, we'll have to build up a decent definition of **volume** — it will involve the limit of the sum of volumes. The 3-D analogue of recangles will be **cylinders**. There will be two main methods for calculating the volume of a solid of rotation: the **disk method** (§6.2) and the **shell method** (§6.3).

The Disk Method:

1. **Volume of a cylinder.**

   Volume of a cylinder of radius r and height h is $\pi r^2 h$ (the area of the base times the height). Or, for a cylinder on its side with radius $f(x)$ and "height" (width) Δx, the volume is

   $$\pi \left(f(x)\right)^2 \Delta x.$$

   Does this look suspiciously familiar?

2. **Cross-section of a solid of rotation.**

   We can always cut a solid of rotation so that the **cross-section** is a circle.

   Previous Example. what is the area of the cross-section taken at $x = 1$? At $x = 2$? At any x between 0 and 4?

   At $x = 1$:

   At $x = 2$:

   At any x:

   The book refers to the area of a cross-section at x as $A(x)$, since it's a function of x (or write A(y) if you're doing a solid rotated about a vertical line–we'll get to this).

3. **Estimating the volume of a solid of rotation.**

   We will use cylindrical "slabs" to approximate the volume.

   Previous Example. Estimate the volume using four cylinders of radii $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

   Recall from above: volume of a cylinder of radius $f(x)$ is $A(x)\Delta x$, where $A(x) = \pi \left(f(x)\right)^2$. For our example, $A(x) = \pi x$ and $\Delta x = 1$. So the volume estimate is

   Notice that we estimated the volume of the solid by using as the radius of each cylinder the height at the *right* endpoint of each interval and then added them up. So the formula was

   $$\sum_{i=1}^{4} A(x_i)\Delta x,$$

where $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$. We could also have used left endpoints, midpoints, etc.

Looking even more familiar?

4. **Finding the exact volume of a solid of rotation.**

   Similar to rectangles, our estimate of the volume will get better the more rectangles we put in. The estimate of the volume using right endpoints and n cylinders is

$$\sum_{i=1}^{n} A(x_i)\Delta x.$$

   Can you guess the formula for the *exact* volume from a to b? It is

$$\lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x,$$

   where $\Delta x = b - an$, as always. Of course, we know an easier way to write this! The volume is

$$\int_a^b A(x)\, dx.$$

   For our example, then, the volume is $\displaystyle\int_0^4 \pi x\, dx =$ (finish)

Summary of the disk method for solids formed by rotating about the x-axis:

- Figure out what the cross-sectional area is (for solids of rotation, the cross-section is always a disk with or without a hole, so the area is

$$A(x) = \pi \cdot (f(x) - g(x))^2$$

  (where $f(x)$ is the "top" function and $g(x)$ is the "bottom" function).

- Find the volume, which is $\int_a^b A(x)\, dx$.

If the axis of rotation is $y = c$ for $c \neq 0$, then you'll need to adjust $A(x)$ to account for this.

The way to do it: radius of larger circle is now $f(x) \pm c$ (where $f(x)$ is the "top" function). Similarly, radius of smaller circle is now $g(x) \pm c$. The "$\pm$" depends on whether c is positive or negative. Hence

$$A(x) = \pi\left(f(x) \pm c\right)^2 - \pi\left(g(x) \pm c\right)^2.$$