Section 6.4 - Work - p. 394 Stewart, 4th Ed.

We have used the integral to find the volume of a solid of rotation. Now we will use the integral for a very different application: calculating how much **work** (in the physics sense) it takes to move something a certain distance.

To do this we need to know

- Newton's Second Law
- The difference between *mass* and *weight*
- Two different systems of units (*English* and *metric*)
- What *work* is.

Newton's Second Law. $F = m \cdot a$, where m is the mass of an object and a is the acceleration on the object. F stands for the *net force* acting on the object.

The difference between *mass* and *weight*.

Mass is a property of matter. It is the same everywhere in the universe. The *kilogram* is the main unit of mass in the metric system. The English system does not generally use a unit of mass!

Weight is a *force*. It is not an inherent property of matter. It is the result of gravity acting on a mass. The *pound* is the main unit of weight in the English system. In the metric system, it is the *Newton*, *not* the kilogram!

Two different systems of units.

Quantity	Metric	English
mass	kilograms (kg)	"slugs" (pounds $\div$ 32 ft./s^2)
weight (force)	Newtons (N)	pounds (lb.)
distance	meters (m)	feet (ft.)
time	seconds (s)	seconds (s)
accel. due to gravity	9.8 m/s^2	32 ft./s^2
work	joules (J) = Newton-meters (N-m)	foot-pounds (ft.-lb.)

Notice the different perspective between the two systems! One focuses on *mass*, the other on *weight*.

Work. If the force acting on an object is constant, then $W = F \cdot d$, where F is the net force acting on the object, and d is the distance the object travels under that force.

If the force acting on an object is *not* constant, such as when the weight varies (example: water leaking out of a bucket, being pumped out of a tank, etc.), then we *estimate* the work done by taking n force measurements, then letting $n \to \infty$ and forming the integral

$$W = \int_a^b f(x)\, dx,$$

where the object moves along the x-axis from a to b, and $f(x)$ is the force at x.

Example 1. (constant force): See Example 1, p. 394.

Part (a) (metric system): 1.2 kg is the *mass* of the book. So its weight is ____________ N (this is the [constant] force acting on the book). Therefore the work done to move it 0.7 m is

________________________ .

Part (b) (English system): 20 lb. is the *weight* of the object. This is already the [constant] force acting on the object. Therefore the work done to move it 6 ft. is

________________________ .

Notice that every quantity given above was in the standard units. **Be careful!** You must convert units to the *standard* units for mass, weight, distance, etc. *before* you solve the problem, or your answer will be incorrect!

Example 2. (variable force): **The Spring Problem.**

Hooke's Law says that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx \qquad (k \text{ measured in N/m})$$

Our job in the spring problem is to use information given in the problem to find the *spring constant* k, then use it to answer the question.

§6.4 #6: A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

1. Convert all units.

   Lengths of spring: From 20 to 30 cm is a distance of ____________ m
   From 20 to 25 cm is a distance of ____________ m

   The force is already in the correct units.

2. Use information given to find k.

   We know $f(0.1) = k \cdot 0.1 = 25$ N. Therefore $k =$ ____________ N/m.

3. Solve the problem.

   The work done is

Sketches of other examples you will find in this section:

Example 3. (variable force): **The Rope Problem.**
§6.4#11: A heavy rope, 50 ft. long, weighs 0.5 lb./ft. and hangs over the edge of a tall building. How much work is done in pulling the rope to the top of the building?

Weight of rope when x ft. of rope have been pulled up is ____________ $(= \frac{1}{2}$ lb. for each foot of rope remaining). So the work done is

Example 4. (variable force): **The Tank Problem.**
How much work is done in pumping the water out of a tank of a certain shape? For these problems we assume that the pump always lifts the water *from the top.*

Please read Example 4 on p. 396 for an explanation of this type of problem in the metric system.
Important! In either system, set your coordinate system *upside-down*, so that $x = 0$ corresponds to the (initial) water level and $x = b$ is a *positive* number which corresponds to the depth of the tank. This will make the coordinate of the pump a *negative* number (or 0 if it is at the initial water level). If you set your coordinates in a different way, you may get an integral which is very hard to evaluate!

For the English system we need to know the weight of water (62.5 lb./ft.^{3}), from which we can calculate the work done as

$$\int_0^b \text{(weight of water)(distance water is to be carried at depth } x\text{)(cross-sectional area) } dx$$

$$= \int_0^b 62.5(x + P)A(x)\, dx,$$

where b is the initial depth of water in the tank and P is the height of the pump above the initial water level. The cross-sectional area becomes part of the integral in the same way it did in §6.2: the volume of a slab of infinitely small thickness Δx is just its cross-sectional area.

More on the metric system:
In the metric system, instead of the weight of water, we are given the *density*, which is $\frac{\text{mass}}{\text{volume}} = 1000$ kg/m^3. You must then find the mass by multiplying by volume:

$$\text{mass} = 1000 \cdot \text{volume}.$$

Finally, we multiply by 9.8 to get the weight. Again, in the integral, the *volume* element will turn into the *cross-sectional area.* Hence the work done is

$$\int_0^b 1000 \cdot 9.8(x + P)A(x)\, dx.$$