## Math 90 Practice Midterm II Solutions

§§4-A – 7-B (Ebersole), 2.1-3.2 (Stewart)

**DISCLAIMER.** This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

**Multiple Choice.** *Circle the letter of the best answer.*

1. The slope of the tangent line to the graph of $f(x) = |x+2|$ at $x = -3$ is

   (a) 1

   (b) $\boxed{-1}$

   (c) 0

   (d) undefined.

   $f(x) = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0. \end{cases}$ In other words, $f(x)$ has slope 1 if $x+2 > 0$ and $-1$ if $x+2 < 0$. Since $x = -3$ satisfies the second case ($-3+2 < 0$), the slope there is $-1$.

2. The function $s(t) = \dfrac{t^2-9}{t+3}$

   (a) is continuous at $t = -3$

   (b) $\boxed{\text{is not continuous at } t = -3.}$

   $t = -3$ is not in the domain of $s(t)$. So $s(t)$ cannot be continuous at $t = -3$.

3. If $H'(2) = \lim_{h\to 0} \dfrac{\frac{3}{\sqrt[4]{2+h}} - \frac{3}{\sqrt[4]{2}}}{h}$, then $H(t)$ could be

   (a) $\boxed{\dfrac{3}{\sqrt[4]{t}}}$

   (b) $-\dfrac{3}{\sqrt[4]{t}}$

   (c) $-\dfrac{3}{\sqrt[4]{t^5}}$

   (d) $12\sqrt[4]{t^3}$

   According to the formula, $H'(2) = \lim_{h\to 0} \dfrac{H(2+h) - H(2)}{h}$. We want this to be equal to $\lim_{h\to 0} \dfrac{\frac{3}{\sqrt[4]{2+h}} - \frac{3}{\sqrt[4]{2}}}{h}$.

   **Guess:** try $H(t) = \dfrac{3}{\sqrt[4]{t}}$, since we see that $2+h$ is being plugged into that pattern in the formula we are given.

**Check:** Using our guess, we get

$$\begin{aligned} H(2+h) &= \frac{3}{\sqrt[4]{2+h}} \\ H(2) &= \frac{3}{\sqrt[4]{2}}. \end{aligned}$$

This is exactly what we wanted to get. So the answer is $\boxed{H(t) = \frac{3}{\sqrt[4]{t}}}$.

4. $\dfrac{3}{5(\sqrt[4]{x+2})^3} + \dfrac{x^2}{3} - \sqrt{5x} =$
   (a) $\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{-2} - 5x^{1/2}$
   (b) $\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{2} - \sqrt{5x}$
   (c) $\boxed{\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{2} - \sqrt{5}x^{1/2}}$
   (d) $\frac{3}{5}(x+2)^{-4/3} + \frac{1}{3}x^{2} - \sqrt{5}x^{1/2}$

Using the rules of exponents, we get

$$\begin{aligned} \frac{3}{5(\sqrt[4]{x+2})^3} + \frac{x^2}{3} - \sqrt{5x} &= \frac{3}{5(x+2)^{3/4}} + \frac{1}{3}x^2 - \sqrt{5}\sqrt{x} \\ &= \frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^2 - \sqrt{5}x^{1/2}. \end{aligned}$$

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Use the following information to answer questions 5 and 6. Let $A(t)$ be the concentration of a certain drug in a patient's bloodstream, measured in g/m$^3$, $t$ minutes after injection. Suppose $A'(30) = 2.4$ and $A'(90) = -1.3$.

5. $A'(30) = 2.4$ means
   (a) After 30 minutes, the concentration of the drug in the patient's bloodstream is 2.4 g/m$^3$
   (b) During the first 30 minutes after being injected, the concentration of the drug in the patient's bloodstream increased by an average of 2.4 g/m$^3$ per minute
   (c) $\boxed{\text{After 30 minutes, the concentration of the drug in the patient's bloodstream is increasing at a rate of 2.4 g/m}^3\text{ per minute}}$
   (d) After 2.4 minutes, the concentration of the drug in the patient's bloodstream has risen by 30 g/m$^3$

$A'(30)$ represents the rate of change of $A(t)$ at $t = 30$. So it is the rate at which the concentration of the drug is changing, measured in g/m$^3$ per minute, at $t = 30$ minutes after the patient was injected.

6. $A'(90)$ is a negative number, which means

   (a) After 90 minutes, there is a negative amount of the drug in the patient's bloodstream

   (b) $\boxed{\text{After 90 minutes, the concentration of the drug in the patient's bloodstream is decreasing}}$

   (c) 1.3 minutes before the injection, the concentration of the drug in the patient's bloodstream was 90 g/m$^3$

   (d) There is a mistake; $A'(90)$ cannot be a negative number

   A negative slope corresponds to a negative rate of change — in other words, a decrease.

7. Suppose $f(x)$ is a function such that $\lim_{x \to 1} f(x) = 2$. Which of the following is *always* true of $f(x)$?

   (a) $f(x)$ is continuous at $x = 1$

   (b) $f(x)$ is continuous on the intervals $(0, 1)$ and $(1, 2)$

   (c) $f(1) = 2$

   (d) $\boxed{\text{None of these.}}$

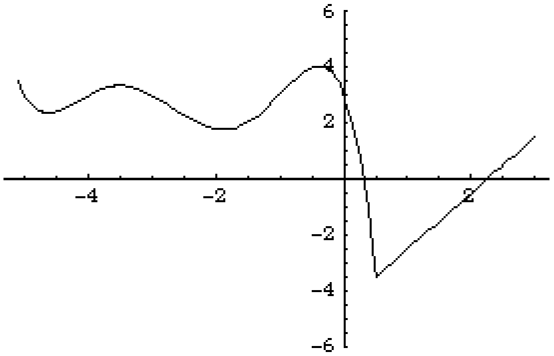
   In order for $f(x)$ to be continuous at $x = 1$, it would have to be true that $\lim_{x \to 1} f(x) = f(1)$. But we are not given any information about whether $f(1)$ is defined or what it might be equal to. So (a) is not always true. And (c) is not always true, for the same reason.

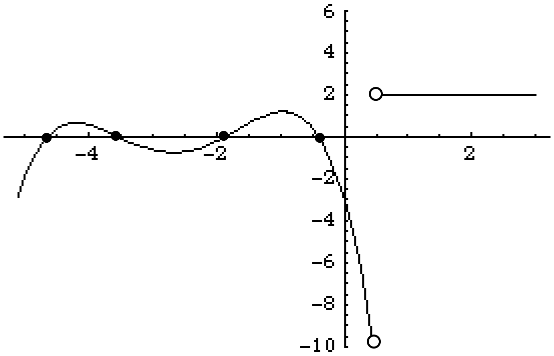
   We are also not given any information about whether $f(x)$ is defined except right near $x = 1$. So we have no guarantee that there isn't a problem at, say, $x = \frac{1}{2}$. So (b) does not always hold, since it is possible to have a vertical asymptote, for example, at $x = \frac{1}{2}$ and still have $\lim_{x \to 1} f(x) = 2$.

## Graph.

For the graph of $f(x)$ shown at right, sketch a graph of $f'(x)$ on the axes below it.

*More accuracy = more points!*

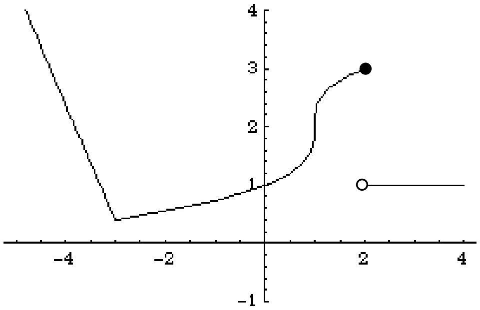




The places where the graph has a horizontal tangent correspond to the zeros on the graph of the derivative (highlighted by dots on the $x$-axis). In between, the slope is either positive (above the $x$-axis) or negative (below the $x$-axis). The derivative of $f(x)$ is undefined at $x = 0.5$, since there is a corner there. Finally, the part of the graph that is a straight line has constant, positive slope approximately equal to 2. So the graph of the derivative shows a horizontal line at $y = 2$.

## Fill-In.

Use the graph of $g(t)$ shown at right to answer questions 1 and 2. For each question, list **all** the $t$-values that make the sentence true.



1. The value(s) of $t$ at which $g(t)$ is not continuous is/are $t = 2$ only .

2. The value(s) of $t$ at which $g'(t)$ is undefined is/are $t = -3$, 1, and 2 .

   $g(t)$ is continuous everywhere except at $t = 2$, since I can draw the graph without lifting my pencil everywhere except there.

   The derivative $g'(t)$ is undefined at corners, vertical tangents, and discontinuous places. There is a corner at $t = -3$, a vertical tangent at $t = 1$, and a discontinuity at $t = 2$.

**Work and Answer.** *You must show all relevant work to receive full credit.*

1. Compute $\lim_{x\to 0^-} \dfrac{|x|-x}{x}$. If the limit does not exist, explain why.

$$\begin{aligned}\frac{|x|-x}{x} &= \begin{cases} \frac{x-x}{x} & \text{if } x>0 \\ \frac{-x-x}{x} & \text{if } x<0 \end{cases} \quad \text{(we write $>$ instead of $\geq$ since $\frac{|x|-x}{x}$ is undefined at $x=0$)} \\ &= \begin{cases} \frac{0}{x} & \text{if } x>0 \\ \frac{-2x}{x} & \text{if } x<0 \end{cases} \\ &= \begin{cases} 0 & \text{if } x>0 \\ -2 & \text{if } x<0. \end{cases}\end{aligned}$$

Since we are only interested in the limit *from the left*, we need only consider the second case above. So $\lim_{x\to 0^-} \dfrac{|x|-x}{x} = \boxed{-2}$ .

2. For the function $g(x) = \frac{2}{x-1}$, compute $g'(-1)$.
*No shortcuts are allowed!*

You can do this problem in two ways, but both of them have to use the formula $\lim_{h\to 0} \ldots$ .
You can use $a=-1$ right away, or you can compute the formula using $a$ and then plug in $-1$ at the end.

**Method 1.** Plug in $-1$ first.

We get

$$\begin{aligned} g'(-1) &= \lim_{h\to 0} \frac{g(-1+h)-g(-1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{(-1+h)-1} - \frac{2}{-1-1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{h-2}+1}{h} \\ &= \lim_{h\to 0} \frac{\frac{2+(h-2)}{h-2}}{h} \\ &= \lim_{h\to 0} \frac{2+h-2}{h(h-2)} \\ &= \lim_{h\to 0} \frac{h}{h(h-2)} \\ &= \lim_{h\to 0} \frac{1}{h-2} = \frac{1}{0-2} = -\frac{1}{2}. \end{aligned}$$

**Method 2.** Get the formula, then plug in $-1$ at the end.

We get

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{(a+h)-1} - \frac{2}{a-1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2(a-1)-2(a+h-1)}{(a+h-1)(a-1)}}{h} \\ &= \lim_{h\to 0} \frac{2a-2-2a-2h+2}{h(a+h-1)(a-1)} \\ &= \lim_{h\to 0} \frac{-2h}{h(a+h-1)(a-1)} \\ &= \lim_{h\to 0} \frac{-2}{(a+h-1)(a-1)} \\ &= \frac{-2}{(a-1)(a-1)} = \frac{-2}{(a-1)^2}. \end{aligned}$$

Then plugging in $a=-1$ we get $g'(-1) = \frac{-2}{(-1-1)^2} = \frac{-2}{4} = -\frac{1}{2}$.

Notice that we got the same answer both ways.

3. Find the equation of the tangent line to the graph of $h(x) = \sqrt{5x}+1$ at $x=4$.

*You may use shortcuts.*

To get the equation of a tangent line, first find the slope. Since we can rewrite $h(x)$ as $\sqrt{5}x^{1/2}+1$, we can use shortcuts to get $h'(x) = \frac{1}{2}\cdot\sqrt{5}x^{-1/2} = \frac{\sqrt{5}}{2\sqrt{x}}$. Therefore the slope at $x=4$ is $h'(4) = \frac{\sqrt{5}}{2\sqrt{4}} = \frac{\sqrt{5}}{4}$.

Next we plug in $x=4$ to the function to get the point of tangency (a point on the graph and also on the tangent line). $h(4) = \sqrt{20}+1 = 2\sqrt{5}+1$. So the point of tangency is $(4, 2\sqrt{5}+1)$.

Now we can use the slope and the point of tangency to get the equation. We get

$$\begin{aligned} 2\sqrt{5}+1 &= \frac{\sqrt{5}}{4}\cdot 4 + b, \qquad \text{so} \\ b &= 2\sqrt{5}+1-\frac{\sqrt{5}}{4}\cdot 4 \\ &= 2\sqrt{5}+1-\sqrt{5} = \sqrt{5}+1. \end{aligned}$$

So the equation of the line is $\boxed{y = \frac{\sqrt{5}}{4}x+\sqrt{5}+1}$.