English system formulas:	Metric system formulas:
1 ft. = 12 in.	$F = m \cdot a$
5280 ft. = 1 mi.	$g = 9.8\,\text{m/s}^2$
16 oz. = 1 lb.	100 cm = 1 m
Weight of water: $\omega = 62.5\,\text{lb./ft}^3$	Weight of water: $\omega = 9800\,\text{N/m}^3$

General formulas:

Hooke's Law: $F(x) = kx$

$$W = \omega \int_0^b (x+P)A(x)\,dx$$

Multiple Choice. *Circle the letter of the best answer.*

1. $\displaystyle\int (\sin x + 2\cos x)\,dx =$

(a) $\boxed{-\cos x + 2\sin x + C}$

(b) $-\cos x - 2\sin x + C$

(c) $\cos x + 2\sin x + C$

(d) $\cos x - 2\sin x$

(e) $-\cos x - 2\sin x$

This is straight out of the formulas.

2. Suppose you know that $f'(x) = g(x)$. Which of the following must be true?

(a) $\displaystyle\int g(x)\,dx = f(x)$

(b) $\boxed{\displaystyle\int g(x)\,dx = f(x) + C}$

(c) $\frac{d}{dx}(g(x)) = f(x)$

(d) $\frac{d}{dx}(g(x)) = f(x) + C$

(e) All of the above are true.

Antidifferentiation is the opposite of differentiation.

3. $\displaystyle\lim_{x\to-\infty} \frac{2x^5 - 3x^3 + 1}{-5x^3 + x - 1} =$

(a) $-\dfrac{2}{5}$

(b) 0

(c) ∞

(d) $\boxed{-\infty}$

(e) does not exist.

Since the biggest power of x is in the numerator, we know the answer must be either ∞ or $-\infty$. To decide which one we use algebra, starting with multiplying the top and bottom by 1 over the biggest power of x occurring in the **denominator**:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{2x^5 - 3x^3 + 1}{-5x^3 + x - 1} &= \lim_{x\to-\infty} \frac{(2x^5 - 3x^3 + 1)\cdot \frac{1}{x^3}}{(-5x^3 + x - 1)\cdot \frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{2x^2 - 3 + \frac{1}{x^3}}{-5 + \frac{1}{x^2} - \frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{2x^2 - 3}{-5} \\
\text{“} = \text{”} & \; \frac{2(-\infty)^2 - 3}{-5} = \boxed{-\infty}
\end{aligned}$$

4. If $x < 0$, then $x^3 =$

   (a) $\sqrt[4]{x^6}$

   (b) $\sqrt[4]{x^8}$

   (c) $-\sqrt[4]{x^8}$

   (d) $\sqrt[4]{x^{12}}$

   (e) $\boxed{-\sqrt[4]{x^{12}}}$

Just a reminder of the “awful truth.” To test whether we need a minus sign for $x < 0$, just plug in -1 and see if the statement holds. For instance, here we have

$$(-1)^3 = -1 \qquad \neq \qquad 1 = \sqrt[4]{(-1)^{12}}.$$

So $x^3 \neq \sqrt[4]{x^{12}}$ for $x < 0$. We need an extra minus sign to make the statement true.

5. If $y = \displaystyle\int_0^{x^2} \tan t \, dt$, then $y' =$

   (a) $\boxed{2x \tan(x^2)}$

   (b) $\tan(x^2)$

   (c) $\tan x$

   (d) $2x \sec^2(x^2)$

   (e) $\sec^2(x^2)$

This is F.T.C. I, combined with the chain rule. The $2x$ is the derivative of the “chunk” x^2.

6. The inflection point(s) of the function $y = 3x^5 - 5x^4 + 60x - 60$ is/are

   (a) $(0, -60)$ only

   (b) $(-1, -128)$ only

   (c) $(-1, -128)$ and $(1, -2)$ only

   (d) $\boxed{(1, -2) \text{ only}}$

   (e) $(0, -60)$, $(1, -2)$, and $(-1, -128)$ only

To get inflection points we take the second derivative and set it equal to 0:

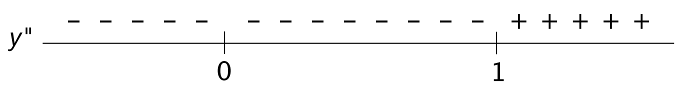
$$\begin{aligned} y' &= 15x^4 - 20x^3 + 60 \\ y'' &= 60x^3 - 60x^2 \overset{\text{set}}{=} 0 \\ 60x^2(x-1) &= 0 \\ x = 0 \quad &, \quad x = 1. \end{aligned}$$

But wait! There are no answer choices that have $x = 0$ and $x = 1$ and nothing else. What gives?

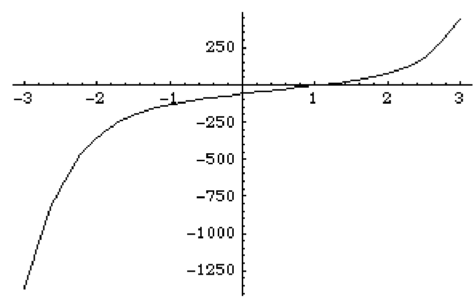
This is a trick question, since **the concavity must change** for a point to be considered an inflection point. So we must test the concavity on either side of 0 and 1.

$$\begin{aligned} y''(-1) &= -60 - 60 < 0 \\ y''\left(\tfrac{1}{2}\right) &= \frac{60}{8} - \frac{60}{4} < 0 \\ y''(2) &= 60 \cdot 8 - 60 \cdot 4 > 0 \end{aligned}$$

These test points give the results shown below.



Looking at the number line we see that the graph is concave down on both sides of 0, i.e. the concavity does not change. So $(0, -60)$ is not an inflection point. Here's what the graph looks like:



7. Which of the following is the linear approximation of the function $f(x) = \sqrt[3]{x}$ near the number $a = 1$?

(a) $y = \frac{1}{3}x + 1$

(b) $\boxed{y = \frac{1}{3}x + \frac{2}{3}}$

(c) $y = x - \frac{2}{3}$

(d) $y = x + 3$

(e) $y = 3x + 2$

This question is just asking for the equation of the tangent line to the graph of $f(x)$ at $x = 1$. So we need the slope and a point on the line. We have $f'(x) = \frac{1}{3}x^{-2/3}$, so the slope at $x = 1$ is $f'(1) = \frac{1}{3}$. Now the point of tangency is $(1, 1)$ since $f(1) = \sqrt[3]{1} = 1$. So using $y = mx + b$ we get

$$\begin{aligned} 1 &= \frac{1}{3} \cdot (1) + b \\ b &= 1 - \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

Therefore the equation of the line is $\boxed{y = \frac{1}{3}x + \frac{2}{3}}$

8. $\displaystyle\int_0^4 |x-3|\,dx =$

(a) 24

(b) 2

(c) 4

(d) 20

(e) $\boxed{5}$

We went over this question in class on Wednesday. Use areas.

9. Let $\mathcal{R}$ be the region enclosed by the lines $y = 2x$, $y = 4x$, and $x = 2$. The volume of the solid formed by rotating $\mathcal{R}$ about the x-axis is

(a) $\displaystyle\pi \int_0^2 \left((2x)^2 - (4x)^2\right)\,dx$

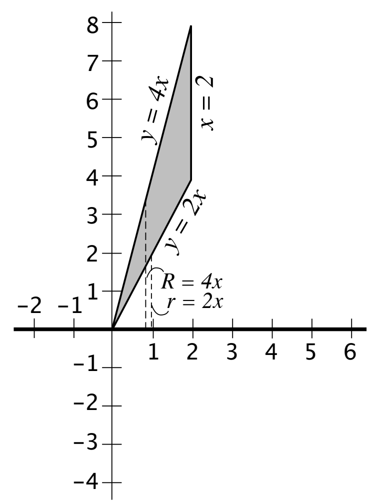
(b) $\displaystyle 2\pi \int_0^8 x(2x-4x)\,dx$

(c) $\displaystyle 2\pi \int_0^8 x(4x-2x)\,dx$

(d) $\boxed{\displaystyle\pi \int_0^2 \left((4x)^2 - (2x)^2\right)\,dx}$

(e) $\displaystyle\pi \int_0^2 (4x-2x)^2\,dx$

Since the region is formed from functions of x and is being rotated about a horizontal line, we use the disk method. The curves make a triangle, as shown.
The correct integral above just follows the formula for disks.



Fill-In.

1. $\displaystyle\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 1}{4x^3 + 2x^2 + x - 1} = \underline{\tfrac{1}{2}}$.

Since the top and bottom have the same degree (biggest power of x), the limit at ∞ is

$$\frac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

You can also use algebra similar to Multiple Choice #3.

2. The vertical asymptote(s) for the function $f(x) = \frac{x}{x^2-1}$ is/are $\underline{x = 1,\ x = -1}$ and the horizontal asymptote(s) is/are $\underline{y = 0}$.

   The denominator is 0 for $x = 1$ and $x = -1$. Since the numerator is not also 0 for these x-values, there is a vertical asymptote at each of these places.

   Since the degree of the bottom is bigger than the degree of the top, we have

$$\lim_{x \to \pm\infty} \frac{x}{x^2 - 1} = 0.$$

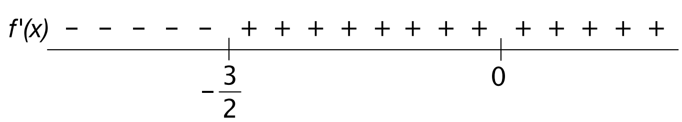
   Therefore there is a horizontal asymptote at $y = 0$.

3. The graph of the function $f(x) = x^4 + 2x^3$ is increasing on the interval(s) $\underline{\left(-\frac{3}{2}, 0\right),\ (0, \infty)}$.

   To check for increasing/decreasing we take the first derivative: $f'(x) = 4x^3 + 6x^2$. First, set it equal to 0 to find the critical numbers:

$$\begin{aligned} 4x^3 + 6x^2 &\overset{\text{set}}{=} 0 \\ 2x^2(2x + 3) &= 0 \\ x = 0 \quad , \quad x &= -\frac{3}{2}. \end{aligned}$$

   Since the domain of $f'(x)$ is all real numbers, there are no "weird" critical numbers (numbers in the domain of $f(x)$ but not in the domain of $f'(x)$). So we set up a number line and check in between the above x-values:



   Looking at the number line we see that the graph is increasing on the intervals $\left(-\frac{3}{2}, 0\right)$ and $(0, \infty)$.

4. According to Rolle's Theorem, the maximum number of real roots of the function $f(x) = 4x^5 + 2x - 3$ is $\underline{\ 1\ }$.

   According to Rolle's Theorem there is at most one more root than the number of solutions to the equation $f'(x) = 0$. We have $f'(x) = 20x^4 + 2 \overset{\text{set}}{=} 0 \Rightarrow$ no solutions! So there is at most 1 real root.

5. Given the initial guess $x_1 = 2$, the second approximation to a root of $g(x) = x^3 - 4x - 1$ using Newton's Method is $x_2 = \underline{\ \frac{17}{8}\ }$.

   We have $g'(x) = 3x^2 - 4$, so $x_2 = 2 - \frac{g(2)}{g'(2)} = 2 - \frac{2^3 - 4 \cdot 2 - 1}{3 \cdot 2^2 - 4} = 2 - \frac{-1}{8} = \frac{17}{8}$.

Graphs. *More accuracy = more points!*

1. For the function $f(x) = \frac{1}{3}x^3 - 2x$,

   (a) find the critical **points** and intervals of increase/decrease

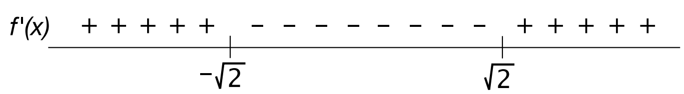
   We have

$$f'(x) = x^2 - 2 \overset{\text{set}}{=} 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

   The domain of $f'(x)$ is all real numbers, so $\sqrt{2}$ and $-\sqrt{2}$ are the only critical numbers. $f(\sqrt{2}) = \frac{1}{3}(\sqrt{2})^3 - 2\sqrt{2} = 2\sqrt{2}\left(\frac{1}{3} - 1\right) = -\frac{4\sqrt{2}}{3}$. $f(x)$ is an odd function (see part (c)), so we know that $f(-\sqrt{2}) = \frac{4\sqrt{2}}{3}$. Therefore the critical points are

$$\left(\sqrt{2}, -\frac{4\sqrt{2}}{3}\right), \left(-\sqrt{2}, \frac{4\sqrt{2}}{3}\right).$$

   Now we set up a number line to find the intervals of increase and decrease:



   $f(x)$ is increasing on $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$ and decreasing on $(-\sqrt{2}, \sqrt{2})$.

   (b) find the inflection **points** and intervals of concave up/concave down

   We repeat the process above for $f''(x)$: We have

$$f''(x) = 2x \overset{\text{set}}{=} 0 \Rightarrow x = 0.$$

   $f(0) = 0$, so $(0, 0)$ is a potential inflection point.
   Since $f''(-1) = -2 < 0$, $f(x)$ is concave down for $x < 0$. Since $f(x)$ is an odd function (see part (c)), we know $f(x)$ is concave up for $x > 0$. Therefore $(0, 0)$ is an inflection point, and $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

   (c) discuss any symmetry $f(x)$ may or may not have

   $f(-x) = \frac{1}{3}(-x)^3 - 2(-x) = -\frac{1}{3}x^3 + 2x = -\left(\frac{1}{3}x^3 - 2x\right) = -f(x)$, so $f(x)$ is an **odd function** (symmetric about the origin).

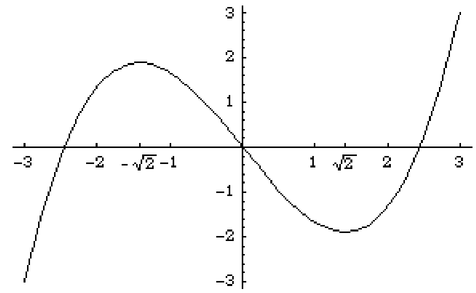
   (d) find the equations of any vertical and/or horizontal asymptotes

   There are no vertical or horizontal asymptotes since $f(x)$ is a polynomial.

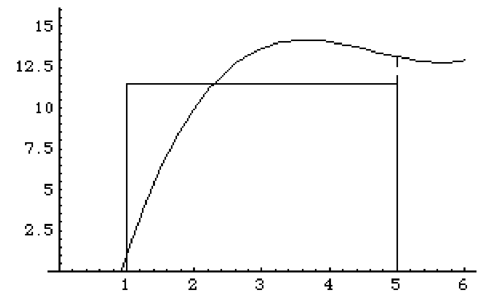
   (e) find the y-intercept

   $f(0) = 0$, so the y-intercept is $(0, 0)$.

(f) On the axes at right, sketch an accurate graph of $f(x)$.



2. (a) For the function $f(x)$ graphed at right, sketch a **rectangle** on the same axes whose area is approximately $\int_1^5 f(x)\,dx$.



(b) The average value f_{ave} of $f(x)$ from $x = 1$ to $x = 5$ is approximately 11.5 .

(c) The approximate value(s) of c so that $f(c) = f_{\text{ave}}$ is/are 2.2 *(list all values)*.

Work and Answer. *You must show all relevant work to receive full credit.*

1. Evaluate $\int_{-1}^{2} (x^2 + 2)\,dx$.

We have

$$\begin{aligned}\int_{-1}^{2} (x^2+2)\,dx &= \frac{1}{3}x^3 + 2x\,\Big|_{-1}^{2} \\ &= \left(\frac{1}{3}\cdot 2^3 + 2\cdot 2\right) - \left(\frac{1}{3}(-1)^3 + 2(-1)\right) \\ &= \frac{8}{3} + 4 + \frac{1}{3} + 2 = 3 + 4 + 2 = \boxed{9}\end{aligned}$$

2. Evaluate $\int x(3x^2+1)^5\,dx$.

Let $u = 3x^2 + 1$. Then $du = 6x\,dx$. Futzing the 6, we get

$$\begin{aligned}\int x(3x^2+1)^5\,dx &= \frac{1}{6}\int 6x(3x^2+1)^5\,dx \\ &= \frac{1}{6}\int u^5\,du \\ &= \frac{1}{6}\cdot\frac{1}{6}u^6 + C \\ &= \boxed{\frac{1}{36}(3x^2+1)^6 + C}\end{aligned}$$

3. A farmer has 400 meters of fencing with which to fence 3 sides of a rectangular horse corral. What is the maximum area she can enclose?

   The objective of this problem is to **maximize the area**. Let x be the width and y the length of the corral. A formula for the area, then, is $A = xy$. We know $2x + y = 400$, so $y = 400 - 2x$. Therefore the area, in terms of x, is

   $$A(x) = x(400 - 2x) = 400x - 2x^2.$$

   Now we find where the absolute maximum of the area function is:

   $$\begin{aligned} A'(x) = 400 - 4x &\overset{\text{set}}{=} 0 \\ 4x &= 400 \\ x &= 100 \end{aligned}$$

   The area is maximized at $x = 100$. The problem asks for the maximum area, so we plug in 100: $A(100) = 100(400 - 2 \cdot 100) = 100(200) = \boxed{20{,}000 \text{ m}^2}$

4. Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = x + 2$.

   First find where the curves intersect:

   $$\begin{aligned} 4 - x^2 &\overset{\text{set}}{=} x + 2 \\ x^2 + x - 2 &= 0 \\ (x - 1)(x + 2) &= 0 \\ x = 1 \quad , \quad x &= -2. \end{aligned}$$

   You should graph the two curves to see which curve is on top. Or plug in any number between -2 and 1, such as 0: $4 - x^2$ comes out more than $x + 2$ ($4 - 0^2 = 4 > 2 = 0 + 2$). So $4 - x^2$ is on top. Therefore the area is

   $$\begin{aligned} \int_{-2}^{1} ((4 - x^2) - (x + 2))\, dx &= \int_{-2}^{1} (2 - x - x^2)\, dx \\ &= 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-2}^{1} \\ &= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \boxed{\frac{9}{2}} \end{aligned}$$

5. Evaluate $\displaystyle\int_0^1 x \cos(x^2 + 1)\, dx$.

   Let $u = x^2 + 1$. Then $du = 2x\, dx$. Also the new limits become

   $$\begin{aligned} 1: &\quad u = 1^2 + 1 = 2 \\ 0: &\quad u = 0^2 + 1 = 1. \end{aligned}$$

Therefore we have

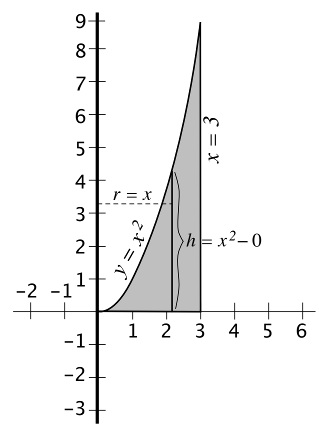
$$\begin{aligned}\int_0^1 x\cos(x^2+1)\,dx &= \frac{1}{2}\int_0^1 2x\cos(x^2+1)\,dx\\ &= \frac{1}{2}\int_1^2 \cos u\,du = \frac{1}{2}\sin u\Big|_1^2\\ &= \boxed{\frac{1}{2}(\sin(2)-\sin(1))}\end{aligned}$$

6. Let $\mathcal{R}$ be the region enclosed by the graphs of $y = x^2$, the x-axis, and $x = 3$. Find the volume of the solid formed by rotating $\mathcal{R}$ about the y-axis.

The region $\mathcal{R}$ is formed with functions of x, and we are rotating about a vertical axis, so we use the **shell method**.

The region is graphed at right. We have $r = x$ and $h = x^2 - 0 = x^2$, so we get

$$\begin{aligned}V &= 2\pi\int_0^3 x\cdot x^2\,dx\\ &= 2\pi\int_0^3 x^3\,dx = 2\pi\cdot\frac{1}{4}x^4\Big|_0^3\\ &= \frac{\pi}{2}\cdot 3^4 = \boxed{\frac{81\pi}{2}}\end{aligned}$$



7. If 9 J of work are required to stretch a spring 75 cm beyond its natural length, find the work done in stretching the spring from 75 cm to 1 m beyond its natural length.

The formula for the work done in stretching a spring 75 cm ($=\frac{3}{4}$ m) beyond its natural length is

$$\int_0^{3/4} kx\,dx = \frac{k}{2}x^2\Big|_0^{3/4} = \frac{k}{2}\left(\frac{3}{4}\right)^2 = \frac{9k}{32}$$

(using Hooke's Law). The problem says that this is equal to 9 J. So we set $\frac{9k}{32} = 9$ and solve for k to get $\frac{k}{32} = 1 \Rightarrow k = 32$. Now we can answer the question in the problem. The work done in stretching the spring from $\frac{3}{4}$ m to 1 m beyond the natural length is

$$\begin{aligned}\int_{3/4}^1 32x\,dx &= 16x^2\Big|_{3/4}^1\\ &= 16\left(1-\left(\frac{3}{4}\right)^2\right)\\ &= 16\left(1-\frac{9}{16}\right) = 16\cdot\frac{7}{16} = \boxed{7\text{ J}}\end{aligned}$$