Math 75B Practice Midterm II Solutions (corrected)
Ch. 18-20 (Ebersole), §§4.10-5.5 (Stewart)

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If the velocity of an object at time t is $v(t) = 4t^2 + 1$ ft./s, then its distance in feet at time t is $s(t) = \frac{4}{3}t^3 + t$. **T** $\boxed{\textbf{F}}$

   The distance might be $s(t) = \frac{4}{3}t^3 + t + 450$. There is no initial condition in the problem. Another way to put this is, the problem does not say "distance *from* somewhere." Maybe $s(t)$ represents the distance from Egypt! Since we don't know, we can't say for sure which antiderivative represents the distance.

2. The function $F(x) = \sin 2x + 52$ is an antiderivative of the function $f(x) = 2\cos 2x$. $\boxed{\textbf{T}}$ **F**

   We can check this by taking the derivative of $F(x)$. We get $F'(x) = \cos 2x \cdot 2 = 2\cos 2x = f(x)$.

3. The function $G(x) = 4x^3$ is an antiderivative of the function $g(x) = x^4 - 2$. **T** $\boxed{\textbf{F}}$

   $G'(x) = 12x^2 \neq g(x)$.

4. $-1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} = \sum_{i=-1}^{5} \frac{i}{i+2}$. **T** $\boxed{\textbf{F}}$

   This is a tricky problem, because the pattern on the left-hand side of the equation is disguised. However, all we have to do is find the sum on the right and see if it simplifies to the sum on the left:

$$\begin{aligned}\sum_{i=-1}^{5} \frac{i}{i+2} &= \frac{-1}{1} + \frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} \\ &= -1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} + \frac{5}{7} \\ &\neq -1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3}.\end{aligned}$$

   There is a term missing on the left-hand side, so the two sides of the equation are not equal.

5. $\sum_{i=2}^{4} \frac{i^2}{2} = \frac{29}{2}$. $\boxed{\textbf{T}}$ **F**

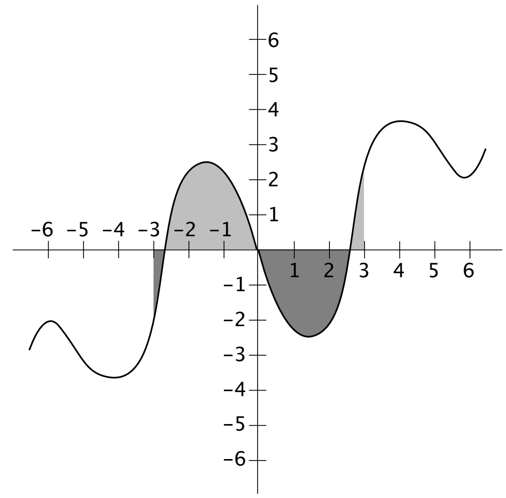
   We have

$$\sum_{i=2}^{4} \frac{i^2}{2} = \frac{2^2}{2} + \frac{3^2}{2} + \frac{4^2}{2} = \frac{4+9+16}{2} = \frac{29}{2}.$$

6. If $g(x)$ is an odd function which is continuous on the interval $[-3, 3]$, **T** **F**

then $\int_{-3}^{3} g(x)\, dx = 0$.

If $g(x)$ is an odd function, that means the graph of $g(x)$ is symmetric about the origin. Therefore between -3 and 3 there is the same amount of area above the x-axis as below. (An example is shown at right.)

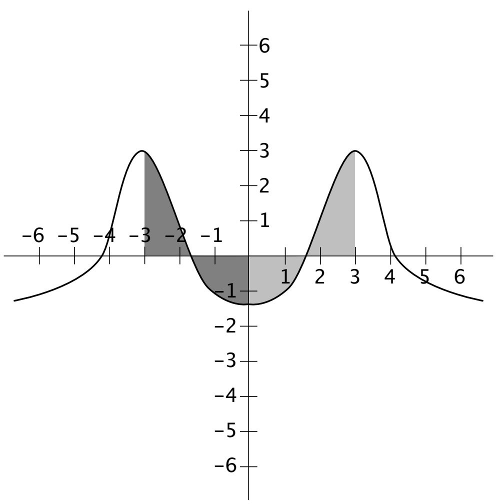


7. If $h(x)$ is an even function which is continuous on the interval $[-3, 3]$, **T** **F**

then $\int_{-3}^{3} h(x)\, dx = 2 \int_{0}^{3} h(x)\, dx$.

This is also true! If $h(x)$ is an even function, that means the graph of $h(x)$ is symmetric about the y-axis. Therefore whatever area there is between 0 and 3 there is the same amount of area between -3 and 0. (An example is shown at right.) So we can just double the amount we get from 0 to 3.

This can be a very useful fact when doing definite integrals, since 0 is a lot easier to plug in than -3.



Multiple Choice. *Circle the letter of the best answer.*

1. $\int_{-2}^{2} \sqrt{4 - x^2}\, dx =$

   (a) $-\frac{1}{6}$
   (b) 0
   (c) $\boxed{2\pi}$
   (d) does not exist.

The curve $y = \sqrt{4 - x^2}$ is the upper half of a circle with radius 2 centered at $(0, 0)$. Therefore the integral $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ represents the area of a semicircle of radius 2, which is $\frac{1}{2} \cdot \pi \cdot 2^2 = \boxed{2\pi}$.

2. $\displaystyle\int_0^{\pi/4} \sec x \tan x \, dx =$

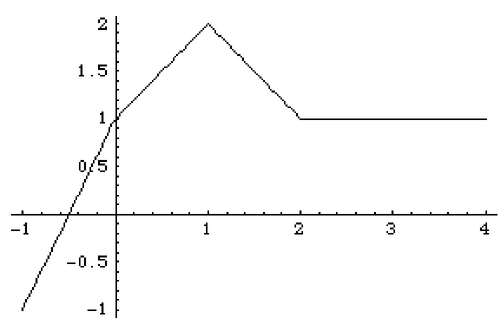
(a) $\boxed{\sqrt{2}-1}$
(b) $\sqrt{2}$
(c) $1-\frac{\sqrt{2}}{2}$
(d) does not exist.

The function $f(x) = \sec x \tan x$ is defined on the interval $\left[0, \frac{\pi}{4}\right]$, so the integral exists. Using the Fundamental Theorem of Calculus we have

$$\int_0^{\pi/4} \sec x \tan x \, dx = \sec x \Big|_0^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0) = \boxed{\sqrt{2}-1}$$

For #3-4, use the graph of $f(x)$ shown below to answer the questions.

3. $\displaystyle\int_{-1}^{0} f(x) \, dx \ =$



(a) -1
(b) $\boxed{0}$
(c) 1
(d) 2

Between $x = -1$ and $x = 0$ there is just as much area (between $f(x)$ and the x-axis) above the x-axis as below. Therefore the net area is $\boxed{0}$.

4. $\displaystyle\int_{2}^{1} f(x) \, dx \ =$

(a) $\boxed{-\frac{3}{2}}$
(b) $\frac{3}{2}$
(c) -1
(d) 1

The area under $f(x)$ from $x = 1$ to $x = 2$ is $\frac{3}{2}$. But the limits of integration have been reversed. So

$$\int_2^1 f(x) \, dx = -\int_1^2 f(x) \, dx = \boxed{-\frac{3}{2}}$$

5. $\displaystyle\int_0^{\pi} \cos\left(5\theta - \frac{\pi}{2}\right)\, d\theta =$

(a) $\boxed{\dfrac{2}{5}}$

(b) $-\dfrac{2}{5}$

(c) $\dfrac{\pi}{2}$

(d) $-\dfrac{\pi}{2}$

Let $u = 5\theta - \frac{\pi}{2}$. Then $du = 5\, d\theta$. We also have that when $\theta = 0$, $u = -\frac{\pi}{2}$ and when $\theta = \pi$, $u = 5\pi - \frac{\pi}{2} = \frac{9\pi}{2}$. So we get

$$\begin{aligned}
\int_0^{\pi} \cos\left(5\theta - \frac{\pi}{2}\right)\, d\theta &= \frac{1}{5}\int_0^{\pi} 5\cos\left(5\theta - \frac{\pi}{2}\right)\, d\theta \\
&= \frac{1}{5}\int_{-\pi/2}^{9\pi/2} \cos u\, du \\
&= \frac{1}{5}\sin u\Big|_{-\pi/2}^{9\pi/2} \\
&= \frac{1}{5}\left(\sin\left(\frac{9\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{1}{5}\left(1 - (-1)\right) = \boxed{\frac{2}{5}}
\end{aligned}$$

Fill-In. *If there is no answer to a question, write "NONE" or "D.N.E.".*

1. $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\, dx = \underline{\tfrac{3}{4}x^{4/3} - \tan x + C}$.

$\sqrt[3]{x} = x^{1/3}$, so $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\, dx = \tfrac{3}{4}x^{4/3} - \tan x + C$ (this is straight out of the formulas in §5.4, p. 351 of Stewart).

2. $\displaystyle\int_{-1}^{2} \sqrt[4]{x}\, dx = \underline{\text{ D.N.E. }}$

$\sqrt[4]{x}$ is undefined for $x < 0$. Therefore the integral does not make sense.

3. $\displaystyle\int_{-1}^{2} \sqrt[3]{x}\, dx = \underline{\tfrac{3}{4}\left(2\sqrt[3]{2} - 1\right)}$

Here we have no domain problems — $\sqrt[3]{x}$ is defined for all x. So we use F.T.C. (Fundamental Theorem of Calculus):

$$\begin{aligned}\int_{-1}^{2} \sqrt[3]{x}\, dx &= \int_{-1}^{2} x^{1/3}\, dx \\ &= \frac{3}{4} x^{4/3} \Big|_{-1}^{2} \\ &= \frac{3}{4} \left(2^{4/3} - (-1)^{4/3}\right) \\ &= \frac{3}{4} \left(2\sqrt[3]{2} - 1\right).\end{aligned}$$

You can also leave your answer as $\frac{3}{4}\left(2^{4/3} - 1\right)$ or anything equal to it. Simplify whenever you can, but *at your own risk!*

4. If $F(x) = \int_{5}^{x} \sqrt{5t - t^4}\, dt$, then $F'(x) = \underline{\sqrt{5x - x^4}}$.

   This is exactly what F.T.C. Part 1 says.

5. If $G(x) = \int_{x}^{\sqrt{\pi}} \cot(6t^2)\, dt$, then $G'(x) = \underline{-\cot(6x^2)}$.

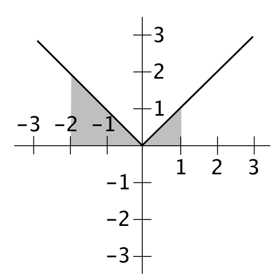
   We have $G(x) = \int_{x}^{\sqrt{\pi}} \cot(6t^2)\, dt = -\int_{\sqrt{\pi}}^{x} \cot(6t^2)\, dt$, so using F.T.C. Part 1 we get the answer shown (the opposite to what we would have gotten if the x had been in the upper limit in the original problem).

6. If the interval $[-4, 7]$ is divided into 6 equal subintervals, then the width of each subinterval is $\underline{\frac{11}{6}}$.

   The interval $[-4, 7]$ is 11 ($= 7 - (-4)$) units wide. If we divide 11 units into 6 equal parts, each part will be $\frac{11}{6}$ units wide.

7. $\int_{-2}^{1} |x|\, dx = \underline{\frac{5}{2}}$.

   The easiest way to do this problem is to think about **areas**. The graph of $|x|$, with the area from -2 to 1 shaded, is shown at right.



   Notice that we have two triangles. The areas of the triangles are $\frac{2\cdot 2}{2} = 2$ and $\frac{1\cdot 1}{2} = \frac{1}{2}$. Since they are both above the x-axis, we add:

$$\int_{-2}^{1} |x|\, dx = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$$

Work and Answer. *You must show all relevant work to receive full credit.*

1. Evaluate $\displaystyle\int \frac{2}{t^3}\,dt$.

   We have

$$\int \frac{2}{t^3}\,dt = \int 2t^{-3}\,dt$$
$$= 2\cdot\frac{1}{-2}t^{-2}+C = -t^{-2}+C = \boxed{-\frac{1}{t^2}+C}$$

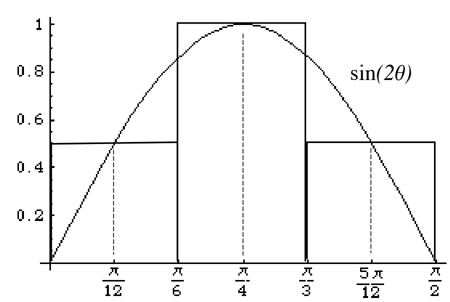
2. Evaluate $\displaystyle\int x^2(5-x^3)^{20}\,dx$.

   Let $u = 5-x^3$. Then $du = -3x^2\,dx$, and we have

$$\int x^2(5-x^3)^{20}\,dx = -\frac{1}{3}\int -3x^2(5-x^3)^{20}\,dx$$
$$= -\frac{1}{3}\int u^{20}\,du$$
$$= -\frac{1}{3}\cdot\frac{1}{21}u^{21}+C$$
$$= \boxed{-\frac{1}{63}(5-x^3)^{21}+C}$$

3. (a) Estimate $\displaystyle\int_0^{\pi/2} \sin 2\theta\,d\theta$ using 3 rectangles and midpoints.

   We are dividing the interval $\left[0,\frac{\pi}{2}\right]$ into 3 equal parts, so each part is of width $\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{3} = \frac{\pi}{6}$. So the three subintervals are $\left[0,\frac{\pi}{6}\right]$, $\left[\frac{\pi}{6},\frac{2\pi}{6}\right]$ and $\left[\frac{2\pi}{6},\frac{\pi}{2}\right]$. The midpoints of these intervals are $\frac{\pi}{12}$, $\frac{3\pi}{12}=\frac{\pi}{4}$, and $\frac{5\pi}{12}$ (see picture at right).



   The heights of the three rectangles, therefore, are

   - $\sin\left(2\cdot\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
   - $\sin\left(2\cdot\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$
   - $\sin\left(2\cdot\frac{5\pi}{12}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

   Therefore the area of the rectangles, put together, is

$$\left(\frac{1}{2}+1+\frac{1}{2}\right)\Delta x = \left(\frac{1}{2}+1+\frac{1}{2}\right)\cdot\frac{\pi}{6}$$
$$= 2\cdot\frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

(b) Evaluate $\int_0^{\pi/2} \sin 2\theta \, d\theta$ exactly.

Let $u = 2\theta$. Then $du = 2\, d\theta$. We use change of variables to convert the θ limits to u limits (you can also "ignore the problem," back-substitute, and use the old limits):

$$\begin{aligned} \theta = \frac{\pi}{2}: &\quad u = 2 \cdot \frac{\pi}{2} = \pi \\ \theta = 0: &\quad u = 2 \cdot 0 = 0 \end{aligned}$$

Therefore we have

$$\begin{aligned} \int_0^{\pi/2} \sin 2\theta \, d\theta &= \frac{1}{2} \int_0^{\pi/2} 2 \sin 2\theta \, d\theta \qquad \text{("futzing" the constant)} \\ &= \frac{1}{2} \int_0^{\pi} \sin u \, du \qquad \text{(change of variables)} \\ &= \frac{1}{2} (-\cos u) \Big|_0^{\pi} \\ &= -\frac{1}{2} (\cos \pi - \cos 0) \qquad \text{(simplifying and using F.T.C.)} \\ &= -\frac{1}{2}(-1-1) = -\frac{-2}{2} = \boxed{1} \end{aligned}$$

(c) What is the error of the estimate you made in part (a)?

The error is $\frac{\pi}{3} - 1 \approx 0.047$, a slight overestimate (π is a little more than 3, so $\frac{\pi}{3}$ is a little more than 1). Looking at the graph and the rectangles, it seems plausible that our estimate would be a little too big, but not by much.

4. If $F(x) = \int_5^{\sin^2 x} (3t - 5) \, dt$,

(a) Evaluate $F(x)$.

We have

$$\begin{aligned} F(x) &= \int_5^{\sin^2 x} (3t-5) \, dt = \frac{3}{2} t^2 - 5t \Big|_5^{\sin^2 x} \\ &= \left(\frac{3}{2} (\sin^2 x)^2 - 5 \sin^2 x \right) - \left(\frac{3}{2} \cdot 5^2 - 5 \cdot 5 \right) \\ &= \boxed{\frac{3}{2} \sin^4 x - 5 \sin^2 x - \frac{75}{2} + 25} \end{aligned}$$

(b) Evaluate $F'(x)$.

Using the chain rule, we get

$$F'(x) = (3 \sin^2 x - 5) \cdot 2 \sin x \cos x = \boxed{6 \sin^3 x \cos x - 10 \sin x \cos x}$$

(c) Show that the derivative of the function you obtained in (a) equals the function you obtained in (b).

Taking the derivative of the function $F(x)$ found in (a), we get

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{2}\sin^4 x - 5\sin^2 x - \frac{75}{2} + 25\right) &= \frac{3}{2}\cdot 4\sin^3 x\cos x - 10\sin x\cos x \\ &= 6\sin^3 x\cos x - 10\sin x\cos x,\end{aligned}$$

which is the answer we got in (b).

5. Evaluate $\displaystyle\int_0^{\pi/3} x^2 - \sin x\, dx$.

The function $f(x) = x^2 - \sin x$ is continuous on the interval $\left[0, \frac{\pi}{3}\right]$, so the integral makes sense. We have

$$\begin{aligned}\int_0^{\pi/3} x^2 - \sin x\, dx &= \frac{1}{3}x^3 + \cos x\,\bigg|_0^{\pi/3} \\ &= \left(\frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \cos\left(\frac{\pi}{3}\right)\right) - (0+1) \\ &= \boxed{\frac{\pi^3}{81} + \frac{1}{2} - 1}\end{aligned}$$

6. Evaluate $\displaystyle\int_1^2 x(3x^2-1)^4\, dx$.

The function $f(x) = x(3x^2-1)^4$ is continuous on the interval $[1, 2]$, so the integral makes sense. We need u-substitution to find the antiderivative. Let $u = 3x^2 - 1$. Then $du = 6x\ dx$. We can either change the limits from x-limits to u-limits ("change of variables" method) or back-substitute and use the original limits ("ignore the problem" method). I will show the latter method (see the solution to Multiple Choice #5 for a demonstration of the change of variables method).

$$\begin{aligned}\int_1^2 x(3x^2-1)^4\, dx &= \frac{1}{6}\int_1^2 6x(3x^2-1)^4\, dx && \text{(“futzing” the constant)} \\ &= \frac{1}{6}\int_?^? u^4\, du && \text{(“ignore” the limits)} \\ &= \frac{1}{6}\cdot\frac{1}{5}u^5\,\bigg|_?^? \\ &= \frac{1}{30}(3x^2-1)^5\,\bigg|_1^2 && \text{(back-substitute and restore old limits)} \\ &= \frac{1}{30}((3\cdot 2^2-1)^5 - (3\cdot 1^2 - 1)^5) \\ &= \boxed{\frac{1}{30}(11^5 - 32)}\end{aligned}$$