CHAPTER 3

# Companion to Limits

## 3-A COMBINATIONS OF FUNCTIONS

The concept of the limit of a function is one of the most important concepts in calculus. Limits are used to assess the behavior of a function $f(x)$ as $x$ *approaches* a number but does not equal the number. In evaluating limits of functions, algebraic expressions often need to be combined or simplified.

### 3-A.1 Algebraic Combinations of Functions

To graph the function $F(x) = \sqrt{x-1} + 5$, it is helpful to look at it as the sum of two functions; $F(x)$ is the sum of $g(x) = \sqrt{x-1}$ and the constant function $h(x) = 5$. This means that at each $x$-coordinate, to get the $y$-coordinate for $F(x)$, we add the two corresponding $y$-coordinates from the functions $g$ and $h$. The graph of $g(x)$ is a half-parabola and $h(x)$ is a horizontal line; these are shown in Figure 3.1.

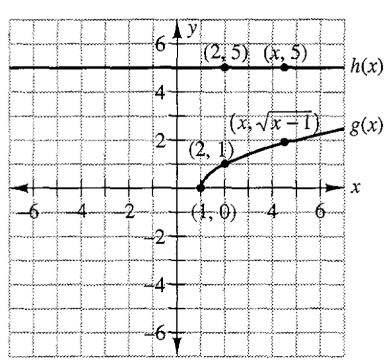



**Figure 3.1**

Adding the constant function $h(x) = 5$ to $g(x)$ adds 5 to each $y$-coordinate of the parabola. This translates the graph up 5 units and produces the graph of $F(x)$ shown in Figure 3.2.

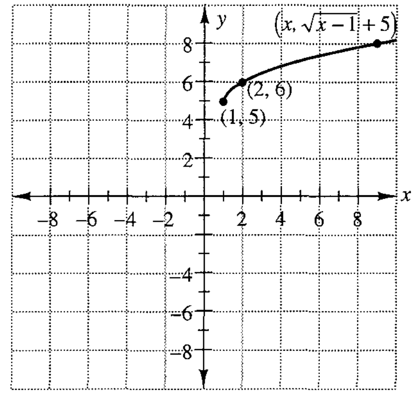



**Figure 3.2** $F(x) = g(x) + h(x)$

We now consider three applications that illustrate how new functions are produced by addition, subtraction, multiplication, and division of functions. Suppose the demand function for a new publication is given by the equation $d = f(p) = 30 - \frac{5}{2}p$, where $p$ is the selling price of the publication measured in dollars and $d$ is the number of copies expected to be sold at that price measured in thousands of copies. This type of demand function is typical: a constant function $g(x) = 30$ (the maximum sales possible) added to a function $h(x) = -\frac{5}{2}p$ that measures how sales drop as price increases.

The marketing department of the publishing company assumes that this function is valid for $2 \le p \le 8$. To estimate the total revenue from the new publication as a function of the price per copy, the product of the demand function (number of copies) times the price per copy is computed. Thus, total revenue is $R(p) = p[30 - \frac{5}{2}p]$, for $2 \le p \le 8$. $R(p)$ is thus a product of two functions.

In another application in business, if $R(x)$ represents the total revenue from the sale of $x$ units of a product and $C(x)$ is the total cost of producing $x$ units, then the total profit $P(x)$ obtained by producing and selling $x$ units is just revenue minus cost:

$$P(x) = R(x) - C(x).$$

The profit function is the difference of the revenue and cost functions. The average cost function is the quotient $\frac{C(x)}{x}$; this measures the average cost of production of 1 unit when $x$ units are produced.

Suppose that $f$ and $g$ are any two functions. Then the sum $f + g$, difference $f - g$, product $f \cdot g$, and quotient $\frac{f}{g}$ are new functions and are defined as follows for each $x$ in the domain of both $f$ and $g$:

**Sum:** $(f + g)(x) = f(x) + g(x)$

**Difference:** $(f-g)(x) = f(x) - g(x)$

**Product:** $(f \cdot g)(x) = f(x) \cdot g(x)$

**Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Each of these defining equations states the rule for the function. For example, the first rule states that the sum function $f+g$ assigns to a number $x$ the sum of the two numbers $f(x)$ and $g(x)$. For any number $x = a$, if either $f(a)$ or $g(a)$ is undefined, then the sum, difference, product, and quotient of $f(a)$ and $g(a)$ are also undefined. In addition, if $g(a) = 0$, then $\frac{f(a)}{g(a)}$ is also undefined. Domains of combinations of functions are discussed more fully in Section 4-C.

**EXAMPLE 3.1** A field is to be enclosed by a fence. The shape of the field is a rectangle capped by two semicircles as shown in Figure 3.3.

The long side of the rectangle is to be twice the length of the short side. Write a function that gives the amount of fence needed to enclose the field as a function of the shorter side of the rectangle.

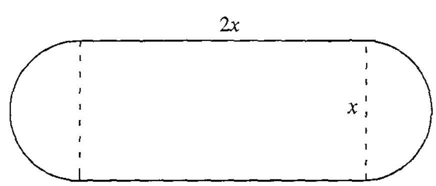



**Figure 3.3**

***Solution*** We label with an $x$ the part of the picture that is to be the independent variable. (Since we want to express the total fencing as a function of the short side of the rectangle, this is labeled $x$.) The amount of fence needed is the sum of the two longer sides of the rectangle and the perimeter of the 2 semicircles (which equal a whole circle). The perimeter of the circle is $C = 2\pi r$, where $r$ is the radius of the circle, but $r = \frac{x}{2}$. Thus $C = 2\pi \frac{x}{2} = \pi x$. Therefore, the amount of fence needed is $F = 2 \cdot 2x + C = 4x + \pi x$. ■

**EXAMPLE 3.2** The supply function $S(x)$ gives the price at which producers will supply an amount $x$ of a product, and the demand function $D(x)$ gives the price at which consumers demand amount $x$ of a product. The equilibrium point is defined to be the point $(x^*, p^*)$, where $x^*$ satisfies $S(x^*) - D(x^*) = 0$ or $S(x^*) = D(x^*)$ (supply = demand) and $p^*$ is the common value of $S(x^*)$ and $D(x^*)$. If the demand function for a certain power tool is $D(x) = 200 - 3x$ and the supply function is $S(x) = 24 + 5x$, find the equilibrium point.

***Solution*** We look at the difference function:

$$S(x) - D(x) = 24 + 5x - (200 - 3x) = -176 + 8x$$

and set this equal to zero. Solving for $x$, we get $8x = 176$, so $x = \frac{176}{8} = 22$. Since $S(22) = 24 + 5(22) = 134$, the equilibrium point is (22, 134). This means that according to the given supply and demand functions, when producers supply 22 of the power tools at a price of \$134, consumers will demand exactly 22 tools. The graphs of $S(x)$ and $D(x)$ cross when $x = 22$ (see Figure 3.4). At a price of less than \$134, demand will exceed supply; if the price is more than \$134, supply will exceed demand. Only at a price of \$134 will demand and supply be equal. The graph of $(S - D)(x) = S(x) - D(x)$ is in Figure 3.5. ■

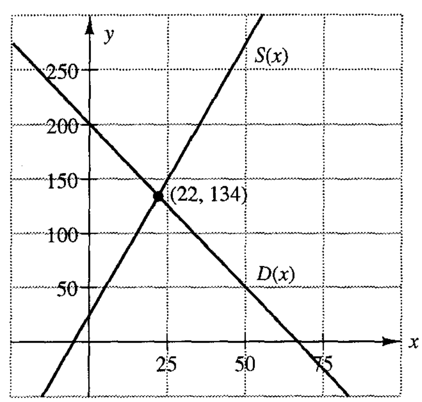



**Figure 3.4**

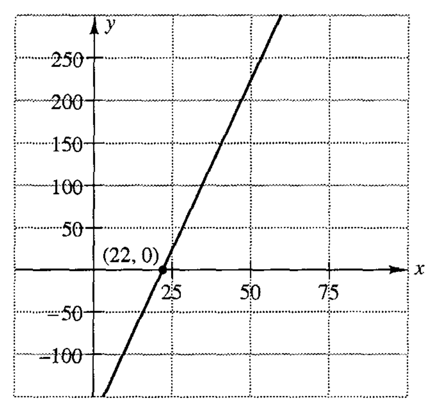



**Figure 3.5** $(S - D)(x) = -176 + 8x$

**EXAMPLE 3.3** Let $f(x) = x^2 - 5x + 6$, and let $g(x) = x^2 - 4$.

**a.** Find $(f + g)(x)$.

**b.** Find $(f \cdot g)(x)$.

**c.** Find $(f - g)(x)$.

**d.** Find $\left(\frac{f}{g}\right)(x)$ and its domain.

***Solution***

**a.** $(f + g)(x) = f(x) + g(x) = (x^2 - 5x + 6) + (x^2 - 4) = 2x^2 - 5x + 2$.

**b.** $(f \cdot g)(x) = f(x) \cdot g(x) = (x^2 - 5x + 6) \cdot (x^2 - 4) = x^4 - 5x^3 + 2x^2 + 20x - 24$.

**c.** $(f - g)(x) = f(x) - g(x) = (x^2 - 5x + 6) - (x^2 - 4) = -5x + 10$.

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 5x + 6}{x^2 - 4}$. The domain of the quotient function is all real numbers that are in the domain of $f$ and in the domain of $g$, excluding those values for which $g(x) = 0$. Since $f(x)$ and $g(x)$ are defined for all real numbers, we only need to exclude values of $x$ for which $x^2 - 4 = 0$. Thus, the domain consists of all real numbers except $x = 2$ and $x = -2$. In interval notation this is $(-\infty,-2) \cup (-2, 2) \cup (2, \infty)$. ■

**EXAMPLE 3.4** Consider the graphs of $y = f(x)$ and $y = g(x)$ shown in Figures 3.6 and 3.7. Estimate the function values indicated in parts a, b, c, and d, if they are defined. If they are undefined, explain why.

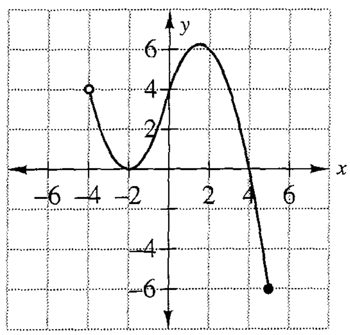



**Figure 3.6** $y = f(x)$

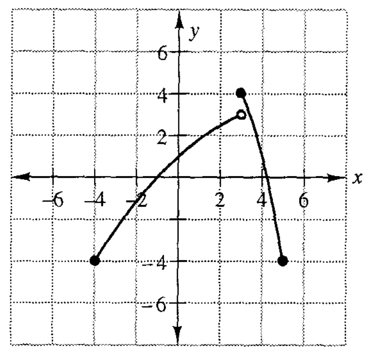



**Figure 3.7** $y = g(x)$

**a.** $(f \cdot g)(0)$  **b.** $\left(\frac{g}{f}\right)(0)$  **c.** $\left(\frac{f}{g}\right)(-1)$  **d.** $(f + g)(3)$

***Solution***

**a.** Since $f(0) = 4$ and $g(0) = 1$, $(f \cdot g)(0) = f(0) \cdot g(0) = 4 \cdot 1 = 4$.

**b.** $\left(\frac{g}{f}\right)(0) = \frac{g(0)}{f(0)} = \frac{1}{4}$.

**c.** $\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)}$ if $g(-1) \neq 0$. But $g(-1) = 0$, so $\left(\frac{f}{g}\right)(-1)$ is undefined.

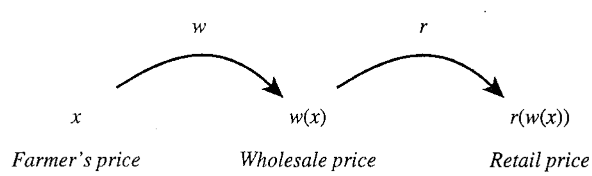
**d.** Since $f(3) = 4$ and $g(3) = 4$, $(f + g)(3) = f(3) + g(3) = 4 + 4 = 8$. ■

## 3-A.2 Composition of Functions

In the previous section we discussed how to combine functions using the four operations of addition, subtraction, multiplication, and division of functions. In this section we will investigate another operation, called *composition,* which combines two functions to produce a new function. Several important theorems in calculus concern the composition of functions.

A function may be interpreted as a rule that produces a value of a dependent variable from a given value of its independent variable. For example, a grocer may use the following rule: Double the wholesale cost of cauliflower to get the retail price. This can be expressed as $r(w) = 2w$, where $w$ is the wholesale price and $r(w)$ is the retail price that corresponds to that wholesale price. But the wholesaler sets his price based on how much he has to pay the farmers for the cauliflower. Suppose he adds \$.15 per head to his cost in setting the wholesale price. This can be expressed as $w(x) = x + 0.15$, where $x$ is the price he pays the farmer per head and $w(x)$ is the corresponding wholesale price. Therefore, if we know that the farmers get \$.35 a head for cauliflower, we can determine the retail price in two steps. First, find the wholesale price $w(0.35)$, which is $0.35 + 0.15$, or $0.50$. Then use the result of this calculation to find the retail price $r(0.50)$, which is $2(0.50)$, or $1.00$. Thus the retail price of the cauliflower is \$1.00 per head.

This two-step operation is called *composition of functions*. The two functions involved are the wholesale price $w(x)$, which is a function of $x$ (the farmer's price), and the retail price $r(w)$, which is a function of the wholesale price $w$. The *composite* of $w$ and $r$ is the function $r(w(x))$, which is a function of $x$. The symbolic expression $r(w(x))$ is read as "$r$ of $w$ of $x$." In some applications a composite function is treated as a two-step process in which the inner function (inside the parentheses) acts first, then the outer function acts on the result of applying the inner function. This process can be diagrammed as follows for this example:




In many cases the two-step composition can be simplified to a single rule. In our example,

$$(1) \quad r(w(x)) = r(x + 0.15) = 2(x + 0.15) = 2x + 0.30$$

is a single rule that is equivalent to the two-step composition. For example, when $x = 0.35$ we get the same wholesale price as before when we use this single rule for $r(w(x))$:

$$r(w(0.35)) = 2(0.35) + 0.30 = 1.00.$$

If the grocer knows how much the wholesaler is paying the farmer per head, he can compute the new price using equation (1). For example, if the farmer charges \$.75 per head, the retail price is computed as $2(0.75) + 0.30 = \$1.80$.

**EXAMPLE 3.5** Compute the following values where the functions $r$ and $w$ are $w(x) = x + 0.15$ and $r(w) = 2w$. Use both the two-step process and the single rule given in equation (1) on the previous page for the composite function.

**a.** $r(w(0.65))$ **b.** $r(w(1.04))$

***Solution***

**a.** $r(w(0.65)) = r(0.65 + 0.15) = r(0.80) = 2(0.80) = 1.60$, so the retail price per head is \$1.60. Alternatively, $r(w(0.65)) = 2(0.65) + 0.30 = 1.60$.

**b.** $r(w(1.04)) = r(1.19) = 2.38$, so the retail price per head is \$2.38. Also, $r(w(1.04)) = 2(1.04) + 0.30 = 2.38$. ■

Since composition is an operation that combines functions, mathematicians have created a special symbol for this operation, $\circ$. The composition of the two functions $f$ and $g$ is denoted $f \circ g$, where $(f \circ g)(x) = f(g(x))$. Using this symbol, our function $r(w(x))$ is written $(r \circ w)(x)$. The product of two functions $f$ and $g$ is written as $f \cdot g$, or, just $fg$.

> ☞ Note that $r \circ w$ does not mean the product of $r$ and $w$.
> $(r \cdot w)(x) = (rw)(x) = r(x)w(x)$ is the product of $r$ and $w$.
> $(r \circ w)(x) = r(w(x))$ is the composite of $r$ and $w$.

Although in our example, $w$ was a function of $x$ and $r$ was a function of $w$, it is common for two or more functions of a single variable $x$ to be composed. In finding the simplified expression for the composite function, the inner function is treated as a single entity and substituted for the variable in the outer function.

**EXAMPLE 3.6** Let $f(x) = x + 2$ and $g(x) = \dfrac{3}{x-1}$.

**a.** Find $(f \circ g)(x)$. **b.** Find $(g \circ f)(x)$.

**c.** Find $(f \circ g)(0)$ and $(g \circ f)(0)$. **d.** What is the domain of $f \circ g$?

**e.** What is the domain of $g \circ f$?

***Solution***

**a.** $(f \circ g)(x) = f(g(x)) = f\left(\frac{3}{x-1}\right) = \frac{3}{x-1} + 2$.

**b.** $(g \circ f)(x) = g(f(x)) = g(x+2) = \frac{3}{(x+2)-1} = \frac{3}{x+1}$.

**c.** $(f \circ g)(0) = \frac{3}{0-1} + 2 = -1$. $(g \circ f)(0) = \frac{3}{(0+2)-1} = 3$.

**EXAMPLE 3.11** Consider the rational function $h(x) = \dfrac{x^3 - x^2 - 3x + 2}{3x^2 - 6x}$.

**a**. Give the domain of $h$.

**b**. Find any common factors of the numerator and denominator, and write $h(x)$ in simplified form.

***Solution***

**a**. The function is not defined at those values of $x$ for which the denominator equals 0. Factoring out the common factor $3x$, the denominator can be written as $3x(x - 2)$, which equals 0 when $x = 0$ and when $x = 2$. The domain is then $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

**b**. We first check to see if $x = 2$ is a root of the numerator. Evaluating the numerator at $x = 2$, we get $2^3 - 2^2 - 6 + 2 = 0$. Since 2 is a root of the numerator, the Factor Theorem tells us that $x - 2$ is a factor. Dividing $x - 2$ into $x^3 - x^2 - 3x + 2$ we can write

$$x^3 - x^2 - 3x + 2 = (x - 2)(x^2 + x - 1).$$

Now we are ready to simplify $h(x)$:

$$\begin{aligned} h(x) &= \frac{x^3 - x^2 - 3x + 2}{3x^2 - 6x} = \frac{(x-2)(x^2 + x - 1)}{3x(x-2)} \\ &= \frac{(x-2)}{(x-2)} \cdot \frac{(x^2 + x - 1)}{3x} = \frac{x^2 + x - 1}{3x} \end{aligned}$$

which is valid only when $x \neq 2$ and $x \neq 0$.

Another way to describe the function $h$ is $h(x) = \dfrac{x^2 + x - 1}{3x}$ if $x \neq 2$ and $x \neq 0$. (The domain of $h(x)$ is all $x \neq 2$, $x \neq 0$.) ■

## 3-B.2 Quotients with Radicals

In the previous section we saw that under certain conditions we could simplify quotients of polynomials by factoring the numerator and denominator polynomials. When the numerator or denominator of a quotient involves a radical, we can change the form of the quotient by *rationalizing* the numerator or denominator; that is, writing the quotient in an equivalent form to remove the radical.

> ☞ Since $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)= a-b,$ $\left(\sqrt{a}-\sqrt{b}\right)$ is rationalized by multiplying by 1 in the form $\frac{\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)}$, and $\left(\sqrt{a}+\sqrt{b}\right)$ is rationalized by multiplying by 1 in the form $\frac{\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}-\sqrt{b}\right)}$.

**EXAMPLE 3.12** Let $f(x)=\sqrt{x}$.

**a**. Give the domain of *f*.

**b**. Write the function $g(x)$ that represents the slope of the secant line to the graph of $y=f(x)$ through the points $\left(x,\sqrt{x}\right)$ and $(9, 3) = \left(9,\sqrt{9}\right)$.

**c**. Give the domain of $g$.

**d**. Write $g(x)$ as a quotient containing no radicals in the numerator, and simplify $g(x)$.

***Solution***

**a**. To find the domain of *f*, we use the fact that the square root is only defined for nonnegative numbers. The domain of *f* is all real numbers $x \geq 0$.

**b**. To write the slope of the secant line through the points $\left(x,\sqrt{x}\right)$ and (9, 3), we use the formula for the slope of the line through two points:

$$g(x) = \text{slope} = \frac{\text{difference in } y\text{-coordinates}}{\text{difference in } x\text{-coordinates}}.$$

Thus, $g(x)=\dfrac{\sqrt{x}-3}{x-9}$. See Figure 3.12 for the graph of $y=f(x)$ and the secant line.

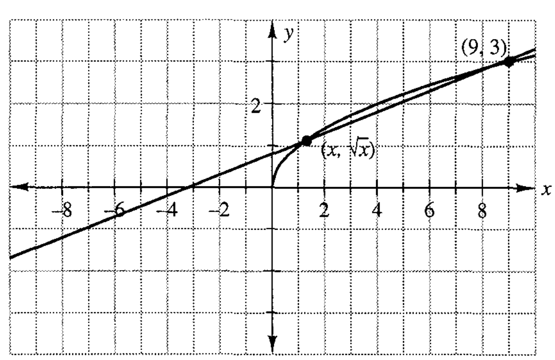



**Figure 3.12**

**c**. To find the domain of $g$, we note that the expression for $g(x)$ involves two operations that are not possible for all real numbers: Square root is only defined for nonnegative numbers, and a quotient is only defined for nonzero denominators. The domain of $g$ then includes every nonnegative number other than $x = 9$. Using interval notation, the domain is described as $[0, 9) \cup (9, \infty)$.

**d**. To rationalize $\sqrt{x} - 3$, we multiply $g(x)$ by $1 = \dfrac{\sqrt{x}+3}{\sqrt{x}+3}$. Then,

$$g(x) = \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \frac{(x-9)}{(x-9)(\sqrt{x}+3)} = \frac{1}{(\sqrt{x}+3)}$$

with this expression for $g(x)$ valid only when $x \neq 9$ and $x \geq 0$. ■

**EXAMPLE 3.13** Simplify each of the following quotients by first rationalizing the numerator.

**a.** $\dfrac{2-\sqrt{x}}{x-4}$  **b.** $\dfrac{\sqrt{4+h}-2}{h}$  **c.** $\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$

***Solution***

**a.**

$$\frac{2-\sqrt{x}}{x-4} = \frac{2-\sqrt{x}}{x-4} \cdot \left(\frac{2+\sqrt{x}}{2+\sqrt{x}}\right) = \frac{(2-\sqrt{x})(2+\sqrt{x})}{(x-4)(2+\sqrt{x})} =$$

$$\frac{4-x}{(x-4)(2+\sqrt{x})} = \frac{-(x-4)}{(x-4)(2+\sqrt{x})} = \frac{-1}{2+\sqrt{x}}, \quad x \neq 4.$$

> ☞ $a - b = -(b - a)$, since $-(b - a) = -b + a = a - b$.

**b.**

$$\frac{\sqrt{4+h}-2}{h} = \frac{(\sqrt{4+h}-2)}{h} \cdot \frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)} =$$

$$\frac{4+h-4}{h(\sqrt{4+h}+2)} = \frac{h}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}, \quad h \neq 0.$$

**c.**

$$\frac{\sqrt{x}-\sqrt{3}}{x-3} = \frac{(\sqrt{x}-\sqrt{3})}{x-3} \cdot \frac{(\sqrt{x}+\sqrt{3})}{(\sqrt{x}+\sqrt{3})} =$$

$$\frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} = \frac{1}{\sqrt{x}+\sqrt{3}}, \quad x \neq 3.$$

■

### 3-B.3 Complex Fractions

You will sometimes need to simplify a quotient that contains a fraction in the numerator or in the denominator. To simplify in these cases, you may need to find a common denominator, as well as use the basic algebra rules for fractions. The next two examples demonstrate this.

| | |
|---|---|
| Some rules of fractions: | **1.** $\dfrac{\left(\frac{a}{b}\right)}{c} = \dfrac{a}{bc}$ <br> **2.** $\dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$ <br> **3.** $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ |

**EXAMPLE 3.14** In a laboratory experiment, a population of 1500 bacteria is placed in a cultured medium and grows in such a way that the number of bacteria after $t$ hours is

$$N(t) = 1500\left(1 + \frac{2t}{30 + t^2}\right).$$

The difference $N(t) - N(0)$ indicates the increase in the number of bacteria during the first $t$ hours. The ratio of that difference to the time elapsed,

$$\frac{N(t) - N(0)}{t}$$

represents the average rate of growth of the bacteria during the first $t$ hours (in number of bacteria per hour).

**a**. Simplify this quotient.

**b**. Find the average rate of growth of the bacteria during the first 10 hours.

***Solution***

**a.**
$$\frac{N(t) - N(0)}{t} = \frac{1500\left(1 + \frac{2t}{30 + t^2}\right) - 1500}{t} =$$

$$\frac{1500 + \frac{3000t}{30 + t^2} - 1500}{t} = \frac{\frac{3000t}{30 + t^2}}{t} = \frac{3000t}{\left(30 + t^2\right)t} = \frac{3000}{30 + t^2}.$$

**b**. Use the simplified form of the quotient $\frac{N(t)-N(0)}{t}$ found in part a, with $t = 10$:

$$\frac{3000}{30+10^2} = \frac{3000}{30+100} = \frac{3000}{130} \approx 23.08 \text{ bacteria per hour.}$$ ■

**EXAMPLE 3.15** Simplify each of the following expressions.

**a.** $\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$ **b.** $\dfrac{(h+5)^{-1}-5^{-1}}{h}$

***Solution*** In both expressions, find a common denominator for the difference of fractions in the numerator. In solving parts a and b we show two different techniques of simplification.

**a.** $$\frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \frac{3x\left(\frac{1}{x}-\frac{1}{3}\right)}{3x(x-3)} = \frac{3-x}{3x(x-3)} = \frac{-(x-3)}{3x(x-3)} = \frac{-1}{3x}, \quad x \neq 3.$$

**b.** $$\frac{(h+5)^{-1}-5^{-1}}{h} = \frac{\frac{1}{h+5}-\frac{1}{5}}{h} = \frac{\frac{5}{5(h+5)}-\frac{(h+5)}{5(h+5)}}{h} = \frac{\frac{5-(h+5)}{5(h+5)}}{h} =$$
$$\frac{5-h-5}{5(h+5)h} = \frac{-h}{5(h+5)h} = \frac{-1}{5(h+5)}, \quad h \neq 0.$$ ■

> ☞ Be careful to keep parentheses when multiplying by a sum such as $h+5$.

## 3-B.4 Quotients with Absolute Values

When considering quotients that involve an absolute value, the definition of absolute value requires that we consider two cases. Recall from Chapter 2 that, for **any** expression $w$,

$$|w| = -(w) \text{ if } (w) < 0, \text{ and } |w| = w \text{ if } (w) \geq 0.$$

Also, $|w| = |-w|$, for all $w$.

**EXAMPLE 3.16** Give the domain and a simplified expression for the following functions $f$ and $g$. Graph each function.

**a.** $f(x) = \dfrac{|x-3|}{x-3}$ **b.** $g(x) = \dfrac{|-2x+2|}{x-1}$

***Solution***

**a**. Since the denominator cannot take the value zero, the domain contains every $x \neq 3$. The domain of $f$ is then $(-\infty, 3) \cup (3, \infty)$. To find a simplified expression, we use the definition of absolute value to write $f$ as a piecewise expression as follows.

If $x - 3 < 0$, then $|x - 3| = -(x - 3)$, so

$$f(x) = \frac{|x-3|}{x-3} = \frac{-(x-3)}{x-3} = -1, \text{ for } x - 3 < 0.$$

If $x - 3 = 0$, then $f(x)$ is undefined.

If $x - 3 > 0$, then $|x - 3| = x - 3$, so

$$f(x) = \frac{|x-3|}{x-3} = \frac{x-3}{x-3} = 1, \text{ for } x - 3 > 0.$$

Thus $f$ can be defined as the step function:

$$f(x) = \begin{cases} -1 & \text{if } x-3<0 \\ 1 & \text{if } x-3>0 \end{cases}$$

or, equivalently, $f(x) = \begin{cases} -1 & \text{if } x<3 \\ 1 & \text{if } x>3. \end{cases}$

See Figure 3.13 for the graph of $y = f(x)$.

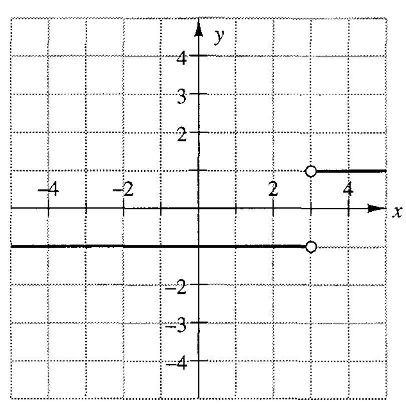



**Figure 3.13** $f(x) = \dfrac{|x-3|}{x-3}$

**b**. The domain of $g$ is the set of all numbers $x$ such that $x - 1 \neq 0$. The domain of $g$ is then $(-\infty, 1) \cup (1, \infty)$.

If $x < 1$ then $-2x + 2 > 0$ and $\dfrac{|-2x+2|}{x-1} = \dfrac{-2x+2}{x-1} = \dfrac{-2(x-1)}{x-1} = -2.$

If $x > 1$ then $-2x + 2 < 0$ and $\dfrac{|-2x+2|}{x-1} = \dfrac{-(-2x+2)}{x-1} = \dfrac{2(x-1)}{x-1} = 2.$

The function $g$ can be defined as $g(x) = \begin{cases} -2 & \text{if } x < 1 \\ 2 & \text{if } x > 1. \end{cases}$

See Figure 3.14 for the graph of $g$. ■

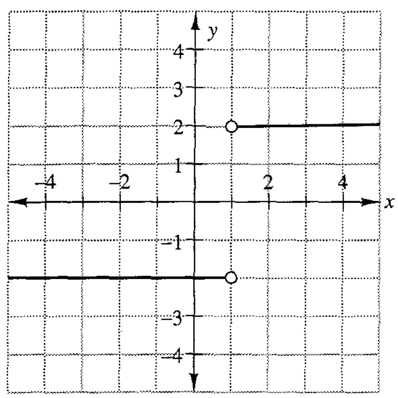



**Figure 3.14** $g(x) = \dfrac{|-2x+2|}{x-1}$

## Exercises 3-B

**1**. Simplify the following functions and give the domain of each function.

**a**. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 6x + 9}$ **b**. $g(x) = \dfrac{\frac{2}{x+3} - \frac{2}{3}}{x}$

**c**. $h(x) = \dfrac{\frac{x}{4} - \frac{1}{2}}{x-2}$ **d**. $v(x) = \dfrac{|-3x+2|}{3x-2}$ **e**. $w(x) = \dfrac{x+3}{x^2-9}$

**2**. **a**. Write the function $m(x)$ that gives the slope of the secant line through the point $\left(1, \frac{1}{3}\right)$ and $\left(x, \frac{1}{x+2}\right)$ on the graph of the function $f(x) = \frac{1}{x+2}$, for $x > -2$.

**b**. Simplify the function $m(x)$ found in part a.

**c**. Sketch the graph of the function $f(x)=\frac{1}{x+2}$, for $x>-2$ and draw the secant line found in part a for several values of $x$.

**3**. **a**. Write the function $n(x)$ that gives the slope of the secant line through the points $\left(2,\frac{1}{4}\right)$ and $\left(x,\frac{1}{x+2}\right)$ on the graph of the function $f(x)=\frac{1}{x+2}$, for $x>-2$.

**b**. Simplify the function $n(x)$ in part a.

**4**. **a**. Complete the following table using $m(x)$ from Exercise 2 and $n(x)$ from Exercise 3.

| $x$ | $-1$ | $0$ | $3$ | $5$ |
|---|---|---|---|---|
| $m(x)$ | | | | |
| $n(x)$ | | | | |

**b**. Explain the values in the table. Are your answers reasonable from the graph?

**5**. Rationalize the numerator in each of the following expressions and simplify:

**a**. $\dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ **b**. $\dfrac{\frac{1}{\sqrt{x}}-\frac{1}{5}}{x-25}$ **c**. $\dfrac{\sqrt{3-h}-\sqrt{3}}{h}$

**d**. $\sqrt{x+1}-\sqrt{x}$ (Note that any expression $w$ can be written as $\frac{w}{1}$, so if $w$ has a square root, we can rationalize the numerator of $w$ by writing it as $\frac{w}{1}$.)

**6**. Sketch the graph of $y=\sqrt{x}$. Locate the point (1, 1) on the graph and sketch secant lines through the points (1, 1) and $\left(x,\sqrt{x}\right)$ for several different choices of $x$. What can you say about the slope of these secant lines?

**7**. Rationalize the denominator in each of the following expressions.

**a**. $\dfrac{x-9}{3-\sqrt{x}}$ **b**. $\dfrac{5-x}{\sqrt{5}-\sqrt{x}}$ **c**. $\dfrac{x-4}{-2+\sqrt{x}}$

**8**. Give the domain and a simplified expression for each of the following functions. Graph each function.

**a**. $f(x)=\dfrac{x^2-7x+10}{x-2}$ **b**. $g(x)=\dfrac{|2-x|}{x-2}$ **c**. $h(x)=\dfrac{x-2}{x^2-7x+10}$

**9**. Let $Q(x)=x^4-a^4$. Factor $Q(x)$ into $x-a$ times a polynomial of degree 3.