Finding the n-th term of a sequence - Section 11.1

To develop a formula for the n-th term of a sequence $\{a_n\}$, look for the following patterns:

(a) **The Arithmetic Pattern.**

If the terms count up by the same number each time, then the n-th term will look like that number times n, plus or minus some correction term.

Example. $\{-4, -1, 2, 5, 8, 11, \ldots\}$.

Notice that each term in the sequence is 3 more than the one before. Therefore the n-th term is $3n \pm \xi$, where the ξ depends on where the sequence starts. If the first term is a_1, then the n-th term is $a_n = 3n - 7$, since a_1 must be -4. If the first term is a_0, then the n-th term is $a_n = 3n - 4$, since a_0 must be -4.

The following are all correct representations of the above sequence:

- $\{3n - 7\}_{n=1}^{\infty}$
- $\{3n - 4\}_{n=0}^{\infty}$
- $\{3n + 2\}_{n=-2}^{\infty}$

(b) **The Power Pattern.**

If the terms are obtained by taking powers of successive integers, then the n-th term will look like $(n \pm \xi)^{\clubsuit}$, where the ξ depends on where the sequence starts.

Example. $\{4, 9, 16, 25, \ldots\}$.

Notice that each term in the sequence is a perfect square, starting with 2^2. Therefore the n-th term is $(n \pm \xi)^2$ for some ξ. If the first term is a_2, then the n-th term is simply $a_n = n^2$, since a_2 must be 4. If the first term is a_1, then the n-th term is $a_n = (n+1)^2$, since ka_1 must be 4.

The following are all correct representations of the above sequence:

- $\{n^2\}_{n=2}^{\infty}$
- $\{(n+1)^2\}_{n=1}^{\infty}$
- $\{(n+2)^2\}_{n=0}^{\infty}$

(c) **The Geometric (Exponential) Pattern.**

If the terms are obtained by *multiplying* by the same number each time, then the n-th term will look like that number to the $(n \pm \xi)$-th power, where the ξ depends on where the sequence starts.

Example. $\{\frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$.

Notice that each term in the sequence is twice the one before. Therefore the n-th term is $2^{n \pm \xi}$ for some ξ. If the first term is a_{-1}, then the n-th term is simply $a_n = 2^n$, since a_{-1} must be $\frac{1}{2}$. If the first term is a_1, then the n-th term is $a_n = 2^{n-2}$, since a_1 must be $\frac{1}{2}$.

The following are all correct representations of the above sequence:

- $\{2^{n-2}\}_{n=1}^{\infty}$
- $\{2^{n+1}\}_{n=-2}^{\infty}$
- $\{2^n\}_{n=-1}^{\infty}$

(d) **The Alternating Pattern.**

If the terms *alternate*, *i.e.* switch from positive to negative every term, then the pattern contains a multiple of $(-1)^{n \pm \xi}$ for some ξ. If $n \pm \xi$ is even, then the n-th term will be positive, so adjust ξ so that the correct terms are positive.

Example. $\{\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, \ldots\}$.

The following are all correct representations of the above sequence:

- $\{(-1)^{n-1}\frac{2}{3}\}_{n=1}^{\infty}$
- $\{(-1)^{n+1}\frac{2}{3}\}_{n=1}^{\infty}$
- $\{(-1)^{n}\frac{2}{3}\}_{n=0}^{\infty}$

(e) **A Combination of Patterns.**

Combine the above techniques, but be careful to adjust everything so that the patterns all start correctly.

Example. $\{0, \frac{2}{3}, -\frac{4}{9}, \frac{6}{27}, -\frac{8}{81}, \ldots\}$.

Observe the following patterns:

(i) Counting by 2's in the numerators
(ii) Powers of 3 in the denominators
(iii) Alternating terms, starting with a negative

Therefore our n-th term will look something like

$$(-1)^{\clubsuit} \frac{2n \pm \heartsuit}{3^{\spadesuit}}.$$

Choose an n to begin with. I'll pick $n = 0$. Then adjust everything so that plugging in $n = 0$ gives $a_0 = 0$:

$$a_n = (-1)^{n+1} \frac{2n}{3^n}.$$

Alternate solution: start with $n = 1$. Then the sequence is

$$\left\{(-1)^n \frac{2n-2}{3^{n-1}}\right\}_{n=1}^{\infty}$$

Notice the following convenient trick:

Convenient Trick. To start the sequence from $n = 1$ instead of $n = 0$, I replaced all the n's in the n-th term formula by $n - 1$. Similarly, if I had wanted to begin with $n = -43$, I could have replaced all the n's in the n-th term formula by $n + 43$:

$$\left\{(-1)^{n+44} \frac{2n+86}{3^{n+43}}\right\}_{n=-43}^{\infty}$$

Practice Problems.

For each problem, find a formula for a_n. If your first term is not a_1, be sure to make it clear what your first term is by writing the sequence in the form $\{a_n\}_{n=?}^{\infty}$.

1. $\{1 \cdot 4,\ 4 \cdot 8,\ 7 \cdot 16,\ 10 \cdot 32,\ \ldots\}$

2. $\{-8, -1, 0, 1, 8, 27, 64, \ldots\}$

3. $\{-\frac{1}{6}, \frac{4}{7}, -2, \frac{64}{9}, -\frac{256}{10}, \ldots\}$