Math 75A Practice Midterm I Solutions
§§2-A – 3-A, Ch. 13, 14 (Ebersole), 1.1-1.5 (Stewart), **W1**

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. A description for the function $f(x) = \sqrt{3x} + 2$ is

   (a) Take 3 times a number and then add 2

   (b) Take 3 times a number, add 2, and then take the square root of the result

   (c) **[Take 3 times a number, take the square root of the result, then add 2]**

   (d) Take $\sqrt{3}$ times a number and then add 2

   $3x$ is under the square root, so we are taking the input and multiplying it by 3, then taking the square root of the result. Finally, we add 2.

2. The range of the function $g(x) = -x^2 + 6x + 5$ is

   (a) $\mathbb{R}$ (all real numbers)

   (b) $[14, \infty)$

   (c) $[-\infty, 14)$

   (d) **[$(-\infty, 14]$]**

   $g(x)$ is a parabola opening down, so the range (outputs) must be from $-\infty$ to the y-coordinate of the vertex. The vertex is at $(3, 14)$ (for a reminder of how to find the vertex of a parabola, see p. 67-68 of Ebersole). Since 14 is in the range, and $-\infty$ is not ($-\infty$ is not a real number!), the range is $(-\infty, 14]$.

3. The graph of the function $g(t) = \sqrt{9 - t^2}$ is

   (a) A circle of radius 9 centered at the origin

   (b) A circle of radius 3 centered at the origin

   (c) The upper half of a circle of radius 9 centered at the origin

   (d) **[The upper half of a circle of radius 3 centered at the origin]**

   $y = \sqrt{r^2 - t^2}$ always represents the upper half of a circle of radius r centered at the origin, since if we square both sides we get $y^2 = r^2 - t^2$, or $t^2 + y^2 = r^2$, which is the equation of a circle of radius r. We get only the upper half because $\sqrt{r^2 - t^2}$ cannot be negative for any input t.

4. $\dfrac{8^t 16^3}{2^t} =$

(a) 2^{3t-12}

(b) $\boxed{2^{2t+12}}$

(c) 2^{12-t}

(d) 2^{8t+3}

We can express everything in the fraction with base 2, as follows: $8 = 2^3$ and $16 = 2^4$, so $8^t 16^3 = (2^3)^t (2^4)^3$, which simplifies to $2^{3t} 2^{4\cdot 3} = 2^{3t+12}$. Finally, we subtract the exponents of the top and bottom to get

$$\frac{2^{3t+12}}{2^t} = 2^{(3t+12)-t} = 2^{2t+12}.$$

5. The inverse of the function $f(x) = 5x^3$ is

(a) $\dfrac{\sqrt[3]{x}}{5}$

(b) $5\sqrt[3]{x}$

(c) $\boxed{\sqrt[3]{\dfrac{x}{5}}}$

(d) $\dfrac{1}{5x^3}$

To compute the inverse of a one-to-one function, switch y and x, then solve for the new y. We have

$$\begin{aligned} x &= 5y^3 \\ y^3 &= \frac{x}{5} \\ y &= \sqrt[3]{\frac{x}{5}}. \end{aligned}$$

So $f^{-1}(x) = \sqrt[3]{\dfrac{x}{5}}$. To check, you can verify that $(f \circ f^{-1})(x) = x$:

$$(f \circ f^{-1})(x) = f\left(\sqrt[3]{\frac{x}{5}}\right) = 5\left(\sqrt[3]{\frac{x}{5}}\right)^3 = 5 \cdot \frac{x}{5} = x.$$

6. If $f(x)$ is a one-to-one function and $f(-3) = 2$ and $f(2) = -5$, then $f^{-1}(2) =$

(a) $\boxed{-3}$

(b) -5

(c) $\dfrac{1}{5}$

(d) $-\dfrac{1}{2}$

The inverse of a function has the x's and y's switched from the original function. In other words, if $f(a) = b$, then $f^{-1}(b) = a$. Here we are told that $f(-3) = 2$, so we know $f^{-1}(2) = -3$.

Fill-In. If $f(x) = 3x - 5$ and $g(x) = e^x$, then

1. $(g \circ f)(1) = \underline{\dfrac{1}{e^2}}$

2. $(g - f)(0) = \underline{\ 6\ }$

3. $(f \circ f)(2) = \underline{\ -2\ }$

4. $(f \circ g)(-1) = \underline{\dfrac{3}{e} - 5}$

5. $f(5) = \underline{\ 10\ }$

6. $f^{-1}(5) = \underline{\dfrac{10}{3}}$

7. $g(2) = \underline{\ e^2\ }$

8. $g^{-1}(2) = \underline{\ \ln(2)\ }$

#5 and #7 are easy, so we'll start with those. We have $f(5) = 3(5) - 5 = 15 - 5 = 10$ and $g(2) = e^2$.

The rest of these problems can be done in two ways. You can either just **plug in the specific inputs** into each function, or you can **compute the formulas** for $(g \circ f)(x)$, $(g - f)(x)$, $(f \circ f)(x)$, etc. and then plug in the values to each new formula.

Method 1.
We have

$$\begin{aligned}
(g \circ f)(1) = g(f(1)) &= g(3(1) - 5) \qquad \text{(since } f(1) = 3(1) - 5\text{)} \\
&= g(-2) = e^{-2} = \boxed{\frac{1}{e^2}} \\
(g - f)(0) = g(0) - f(0) &= e^0 - (0 - 5) \\
&= \boxed{6} \\
(f \circ f)(2) = f(f(2)) &= f(3(2) - 5) \qquad \text{(since } f(2) = 3(2) - 5\text{)} \\
&= f(1) = \boxed{-2} \\
(f \circ g)(-1) = f(g(-1)) &= f\left(\frac{1}{e}\right) \qquad \text{(since } g(-1) = e^{-1} = \frac{1}{e}\text{)} \\
&= 3 \cdot \frac{1}{e} - 5 = \boxed{\frac{3}{e} - 5}
\end{aligned}$$

For #6 we must ask: *What do we have to plug in to $f(x)$ to get 5 out?* So set $f(x) = 5$ and solve for x; we have

$$\begin{aligned}
3x - 5 &\overset{\text{set}}{=} 5 \\
3x &= 10 \\
x &= \frac{10}{3}
\end{aligned}$$

Therefore $f^{-1}(5) = \boxed{\frac{10}{3}}$

Similarly, on #8 we set $g(x) = 2$ and solve for x:

$$\begin{aligned}
e^x &= 2 \\
x &= \ln(2)
\end{aligned}$$

Therefore $g^{-1}(2) = \boxed{\ln(2)}$

Note. You will not be responsible for logarithms on this exam. :-)

Method 2.
First we have

$$(g \circ f)(x) = g(f(x)) = g(3x-5) = e^{3x-5}$$
$$(g - f)(x) = e^x - (3x-5) = e^x - 3x + 5$$
$$(f \circ f)(x) = f(f(x)) = f(3x-5) = 3(3x-5) - 5 = 9x - 15 - 5 = 9x - 20$$
$$(f \circ g)(x) = f(g(x)) = f(e^x) = 3e^x - 5.$$

Using the first formula we get $(g \circ f)(1) = e^{3(1)-5} = \boxed{e^{-2}}$

Using the second formula we get $(g - f)(0) = e^0 - 3(0) + 5 = \boxed{6}$

Using the third formula we get $(f \circ f)(2) = 9(2) - 20 = 18 - 20 = \boxed{-2}$

Using the fourth formula we get $3 \cdot \frac{1}{e} - 5 = \boxed{\frac{3}{e} - 5}$

For #6 we compute $f^{-1}(x)$ by the "switch y and x" trick discussed in class. We have

$$y = 3x - 5$$
$$x = 3y - 5$$
$$3y = x + 5 y = \frac{1}{3}(x+5)$$

So $f^{-1}(x) = \frac{1}{3}(x+5)$. Therefore $f^{-1}(5) = \frac{1}{3}(5+5) = \boxed{\frac{10}{3}}$

Similarly for #8 we have

$$y = e^x$$
$$x = e^y$$
$$y = \ln(x)$$

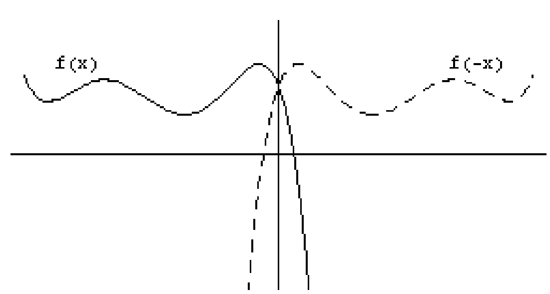
So $g^{-1}(2) = \boxed{\ln(2)}$

Note. You will not be responsible for logarithms on this exam. :-)

Graphs.

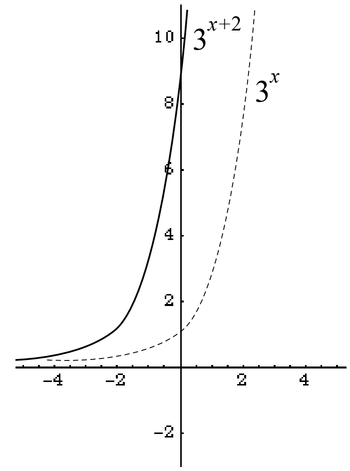
1. The graph of $f(x)$ is shown at right.
   On the same axes, sketch the graph of $f(-x)$.

   The graph of $f(-x)$ is shown with dashed lines. Notice that it is the **(horizontal)** reflection of $f(x)$ about the y-axis.



2. On the axes below, sketch the graph of $f(x) = 3^{x+2}$.

   The graph of $f(x) = 3^{x+2}$ is obtained by shifting that of $g(x) = 3^x$ to the left 2 units. Both graphs are shown at right.



3. On the axes below, sketch the graph of $h(x) = 2|x-1| + 3$.

   There are two ways to do this problem:

   (1) Transformations of $|x|$. Notice that if we perform the following transformations, we will get $h(x)$:

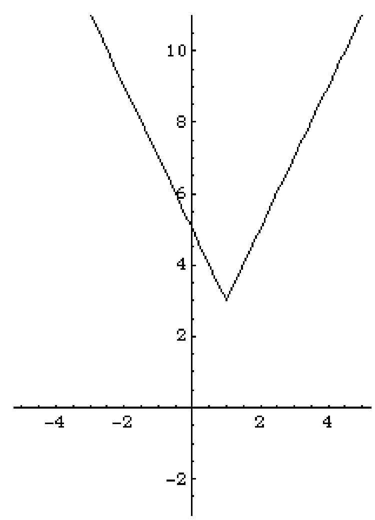
$$|x| \xrightarrow[\text{right 1}]{\text{shift}} |x-1| \xrightarrow[\text{vertically}]{\text{stretch}} 2|x-1| \xrightarrow[\text{up 3}]{\text{shift}} 2|x-1|+3.$$

   Therefore the graph looks like the picture at right.

   (2) Piecewise function.

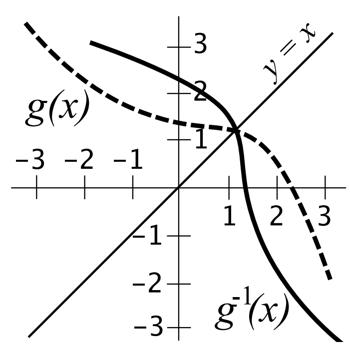
$$h(x) = \begin{cases} 2(x-1)+3 & \text{if } x-1 \geq 0 \\ -2(x-1)+3 & \text{if } x-1 < 0 \end{cases}$$
$$= \begin{cases} 2x+1 & \text{if } x \geq 1 \\ -2x+5 & \text{if } x < 1. \end{cases}$$

   $h(1) = 2(1)+1 = 3$, so the vertex is at $(1,3)$, and we get the graph shown.



4. The graph of $g(x)$ is shown at right.
   On the same axes, sketch the graph of $g^{-1}(x)$.

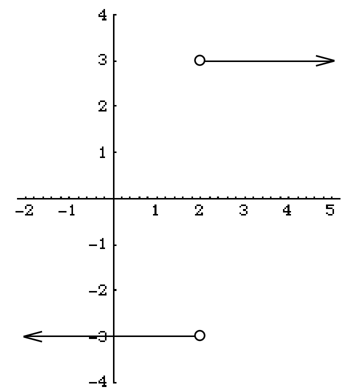
   To get the graph of the inverse of a function, reflect the graph about the line $y = x$. The line $y = x$ is shown for reference.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Write $f(x) = \dfrac{|3x-6|}{x-2}$ as a piecewise function and graph the function. What is the domain of $f(x)$?

$$f(x) = \begin{cases} \dfrac{3x-6}{x-2} & \text{if } 3x-6>0 \\ \dfrac{-(3x-6)}{x-2} & \text{if } 3x-6<0 \end{cases}$$
$$= \begin{cases} \dfrac{3(x-2)}{x-2} & \text{if } 3x>6 \\ \dfrac{-3(x-2)}{x-2} & \text{if } 3x<6 \end{cases}$$
$$= \begin{cases} 3 & \text{if } x>2 \\ -3 & \text{if } x<2. \end{cases}$$



Notice that the first case says ">" rather than "≥" since $f(x)$ is undefined at $x=2$. In fact, the domain of $f(x)$ is $\{x \mid x \neq 2\}$.

From the piecewise function, we can see that the graph looks like the one shown.

2. Find the inverse of the function $f(x) = \dfrac{3}{2x-5}$.

To compute the inverse of a one-to-one function, switch y and x, then solve for the new y. We have

$$x = \frac{3}{2y-5}$$

Multiply both sides by $2y-5$:

$$x(2y-5) = 3$$

Divide both sides by x:

$$2y-5 = \frac{3}{x}$$
$$2y = \frac{3}{x} + 5$$
$$\boxed{y = \frac{\frac{3}{x}+5}{2}} \text{ or } \boxed{y = \frac{3}{2x} + \frac{5}{2}}$$

3. Find the domain of the function $g(x) = \dfrac{3\sqrt{x}}{4x-1}$. *Express your answer in interval notation.*

Because of the denominator, $4x - 1 \neq 0$. Solving for x, we get $4x \neq 1$, or $x \neq \dfrac{1}{4}$.

We also have, because of the square root, $x \geq 0$. So the domain is $x \geq 0$ but $x \neq \dfrac{1}{4}$. In interval notation this is $\boxed{\left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)}$