Math 75A Practice Midterm III Solutions

Ch. 6-8 (Ebersole), §§2.7-3.4 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $g(x) = 3x^4 \sin x$, then $g'(x) = 12x^3 \cos x$. **T** [✓ **F**]

   The above statement is the "bad" way to do the product rule! The correct product rule gives $g'(x) = 3x^4 \cos x + 12x^3 \sin x$.

2. $\sec\theta \tan\theta = \dfrac{\sin\theta}{\cos^2\theta}$ for all angles θ. [✓ **T**] **F**

   Notice that $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. So $\sec\theta\tan\theta = \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\cos^2\theta}$.

3. $\sin(5t) = 5\sin t$ for all angles t. **T** [✓ **F**]

   This is almost *never* true! For example, if you plug in $t = \dfrac{\pi}{2}$, on the left hand side you get $\sin\left(\dfrac{5\pi}{2}\right) = 1$, whereas on the right hand side you get $5\sin\left(\dfrac{\pi}{2}\right) = 5 \cdot 1 = 5$. So they are not the same!

4. $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$. [✓ **T**] **F**

   The reference angle for $\dfrac{2\pi}{3}$ (the angle in quadrant I corresponding to $\dfrac{2\pi}{3}$) is $\dfrac{\pi}{3}$. Using our special triangle we get $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$. Since $\dfrac{2\pi}{3}$ is in quadrant II, and the tangent function is always negative in quadrant II, we get $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$.

5. The only solution to the equation $\cos t = -1$ is $t = \pi$. **T** [✓ **F**]

   Any angle that is coterminal with π also satisfies the equation. For example, $t = -\pi$, $t = \pm 3\pi$, $t = \pm 5\pi$, etc.

Multiple Choice. *Circle the letter of the best answer.*

1. If $H'(2) = \lim\limits_{h\to 0} \dfrac{\frac{3}{\sqrt[4]{2+h}} - \frac{3}{\sqrt[4]{2}}}{h}$, then $H(t)$ could be

   (a) $\boxed{\dfrac{3}{\sqrt[4]{t}}}$

   (b) $-\dfrac{3}{\sqrt[4]{t}}$

   (c) $-\dfrac{3}{\sqrt[4]{t^5}}$

   (d) $12\sqrt[4]{t^3}$

According to the formula, $H'(2) = \lim_{h \to 0} \dfrac{H(2+h) - H(2)}{h}$. We want this to be equal to $\lim_{h \to 0} \dfrac{\frac{3}{\sqrt[4]{2+h}} - \frac{3}{\sqrt[4]{2}}}{h}$.

Guess: try $H(t) = \dfrac{3}{\sqrt[4]{t}}$, since we see that $2+h$ is being plugged into that pattern in the formula we are given.

Check: Using our guess, we get

$$\begin{aligned} H(2+h) &= \frac{3}{\sqrt[4]{2+h}} \\ H(2) &= \frac{3}{\sqrt[4]{2}}. \end{aligned}$$

This is exactly what we wanted to get. So the answer is $\boxed{H(t) = \dfrac{3}{\sqrt[4]{t}}}$.

2. $\dfrac{3}{5(\sqrt[4]{x+2})^3} + \dfrac{x^2}{3} - \sqrt{5x} =$

(a) $\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{-2} - 5x^{1/2}$

(b) $\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{2} - \sqrt{5}x$

(c) $\boxed{\frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^{2} - \sqrt{5}x^{1/2}}$

(d) $\frac{3}{5}(x+2)^{-4/3} + \frac{1}{3}x^{2} - \sqrt{5}x^{1/2}$

Using the rules of exponents, we get

$$\begin{aligned} \frac{3}{5(\sqrt[4]{x+2})^3} + \frac{x^2}{3} - \sqrt{5x} &= \frac{3}{5(x+2)^{3/4}} + \frac{1}{3}x^2 - \sqrt{5}\sqrt{x} \\ &= \frac{3}{5}(x+2)^{-3/4} + \frac{1}{3}x^2 - \sqrt{5}x^{1/2}. \end{aligned}$$

Use the following information to answer questions 3 and 4. Let $A(t)$ be the concentration of a certain drug in a patient's bloodstream, measured in g/m^3, t minutes after injection. Suppose $A'(30) = 2.4$ and $A'(90) = -1.3$.

3. $A'(30) = 2.4$ means

   (a) After 30 minutes, the concentration of the drug in the patient's bloodstream is 2.4 g/m^3

   (b) During the first 30 minutes after being injected, the concentration of the drug in the patient's bloodstream increased by an average of 2.4 g/m^3 per minute

   (c) After 30 minutes, the concentration of the drug in the patient's bloodstream is increasing at a rate of 2.4 g/m^3 per minute

   (d) After 2.4 minutes, the concentration of the drug in the patient's bloodstream has risen by 30 g/m^3

$A'(30)$ represents the rate of change of $A(t)$ at $t = 30$. So it is the rate at which the concentration of the drug is changing, measured in g/m^3 per minute, at $t = 30$ minutes after the patient was injected.

4. $A'(90)$ is a negative number, which means

   (a) After 90 minutes, there is a negative amount of the drug in the patient's bloodstream

   (b) $\boxed{\text{After 90 minutes, the concentration of the drug in the patient's bloodstream is decreasing}}$

   (c) 1.3 minutes before the injection, the concentration of the drug in the patient's bloodstream was 90 g/m^3

   (d) There is a mistake; $A'(90)$ cannot be a negative number

   A negative slope corresponds to a negative rate of change — in other words, a decrease.

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5. If $f(x) = 4x^7 - \dfrac{x^2}{5} + 2$, then $f'(x) =$

   (a) $28x^5 - \dfrac{2}{5}x$

   (b) $\boxed{28x^6 - \dfrac{2}{5}x}$

   (c) $4x^7 - \dfrac{1}{5}x^2 + 2$

   (d) $28x^6 - \dfrac{1}{5}x + 2$

   Since $\dfrac{x^2}{5} = \dfrac{1}{5}x^2$, using the power rule with the sum and difference rule we get the answer shown. Note that the derivative of a constant is 0, so the "+2" goes away in the answer.

6. If $g(x) = 6\sqrt[3]{x}$, then $g'(x) =$

   (a) $\boxed{\dfrac{2}{x^{2/3}}}$

   (b) $\dfrac{6}{x^{-1/3}}$

   (c) $6\sqrt[3]{1}$

   (d) $\dfrac{2}{\sqrt[3]{x}}$

   $g(x) = 6x^{1/3}$, so using the power rule we get $g'(x) = 2x^{-2/3} = \dfrac{2}{x^{2/3}}$.

7. If $f(x) = \tan x$, then $f'(x) =$

   (a) $\boxed{\sec^2 x}$

   (b) $\dfrac{\sin x}{\cos x}$

   (c) $\dfrac{1}{\tan x}$

   (d) $\sec x \tan x$

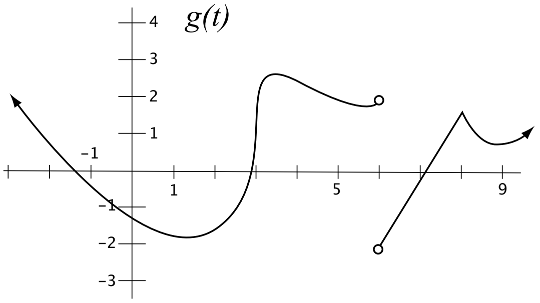
   This is one of the formulas you should memorize. However, if you forget it you can remember that $\tan x = \dfrac{\sin x}{\cos x}$ and use the quotient rule to get the derivative, as we did in class.

Fill-In.

1. For the graph of $g(t)$ shown at right, the value(s) of t at which $g'(t)$ is undefined is/are

   3, 6, and 8 .

   At $t = 3$ the graph has a vertical tangent, so the slope is undefined. At $t = 6$ $g(t)$ is not continuous, so $g'(6)$ cannot be defined. Finally, at $t = 8$ there is a corner, so the derivative is not defined there either.



2. $\sin\left(\dfrac{3\pi}{2}\right) = \underline{\ -1\ }$

   The terminal side of the angle $\dfrac{3\pi}{2}$ points straight down, so it intersects the unit circle at the point $(0, -1)$. $\sin\left(\dfrac{3\pi}{2}\right)$ is the y-coordinate of this point, so the answer is -1.

3. $\cos\left(\dfrac{3\pi}{4}\right) = \underline{\ -\dfrac{\sqrt{2}}{2}\ }$

   The reference angle is $\dfrac{\pi}{4}$. $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$. $\dfrac{3\pi}{4}$ is in quadrant II, so $\cos\left(\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$.

4. $\tan\left(\dfrac{11\pi}{6}\right) = \underline{\ -\dfrac{\sqrt{3}}{3}\ }$

   The reference angle is $\dfrac{\pi}{6}$. $\tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. $\dfrac{11\pi}{6}$ is in quadrant IV, so $\tan\left(\dfrac{11\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$.

5. $\sec\left(\dfrac{17\pi}{3}\right) = \underline{\ 2\ }$

   The reference angle is $\dfrac{\pi}{3}$. $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$, so $\sec\left(\dfrac{\pi}{3}\right) = 2$. $\dfrac{17\pi}{3}$ is in quadrant IV, so $\sec\left(\dfrac{17\pi}{3}\right) = 2$.

6. If $\cos\theta = -\dfrac{1}{5}$ and θ is in quadrant II, then

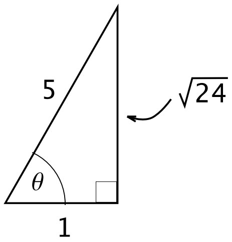
   (a) $\sin\theta = \underline{\ \dfrac{\sqrt{24}}{5}\ }$

   (b) $\tan\theta = \underline{\ -\sqrt{24}\ }$

   (c) $\sec\theta = \underline{\ -5\ }$

   (d) $\csc\theta = \underline{\ \dfrac{5}{\sqrt{24}}\ }$

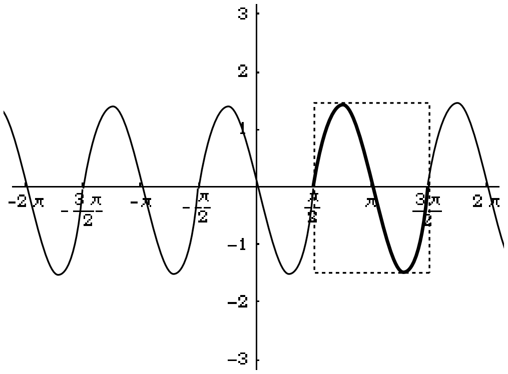
   (e) $\cot\theta = \underline{\ -\dfrac{1}{\sqrt{24}}\ }$



We can "pretend" that θ is in quadrant I and draw a triangle as shown. Notice that the cosine of this "pretend" angle (reference angle) is $\frac{1}{5}$. We can get the third side ($\sqrt{24}$) using the Pythagorean Theorem. Then we find all the trigonometric functions for the reference angle. Finally, since the "real" θ is in quadrant II, we put minus signs on the tangent, secant, and cotangent functions.

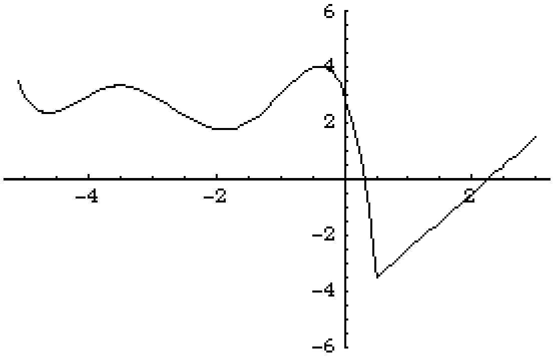
Graphs. *More accuracy = more points!*

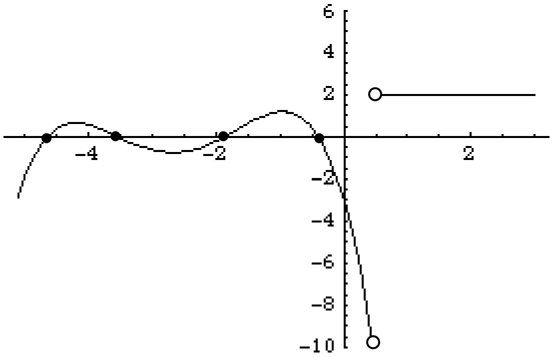
1. On the axes at right, sketch a graph of at least one period of the function $f(t) = \frac{3}{2}\sin(2t - \pi)$.



The first thing to do is to rewrite the function as $f(t) = \frac{3}{2}\sin\left(2\left(t - \frac{\pi}{2}\right)\right)$, so that we can see that the phase shift (horizontal shift) is $\frac{\pi}{2}$ to the right. The box (dashed lines) shows where one period of the function goes. Notice that the amplitude is $A = \frac{3}{2}$, the period is $\frac{2\pi}{2} = \pi$, and there is no vertical shift.

2. For the graph of $f(x)$ shown at right, sketch a graph of $f'(x)$ on the axes below it.





The places where the graph has a horizontal tangent correspond to the zeros on the graph of the derivative (highlighted by dots on the x-axis). In between, the slope is either positive (above the x-axis) or negative (below the x-axis). The derivative of $f(x)$ is undefined at $x = 0.5$, since there is a corner there. Finally, the part of the graph that is a straight line has constant, positive slope approximately equal to 2. So the graph of the derivative shows a horizontal line at $y = 2$.

Work and Answer. *You must show all relevant work to receive full credit.*

1. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, then its height (in meters) after t seconds is $s(t) = 10t - 0.83t^2$.

   (a) What is the velocity of the stone after 3 seconds?

   To get the velocity, we take the derivative of the distance: $s'(t) = 10 - 1.66t$. Then $s'(3) = 10 - 1.66(3) = \boxed{5.02 \text{ m/s}}$.

(b) When does the stone reach its maximum height?

When the stone is at its maximum height, the velocity is 0. So we may get the answer by setting the velocity equal to 0 and solving for t:

$$\begin{aligned} 10 - 1.66t &\overset{\text{set}}{=} 0 \\ 1.66t &= 10 \\ t = \frac{10}{1.66} = \frac{1000}{166} &= \boxed{\frac{500}{83} \approx 6 \text{ seconds}} \end{aligned}$$

2. For the function $g(x) = \dfrac{2}{x-1}$, compute $g'(-1)$.

No shortcuts are allowed!

You can do this problem in two ways, but both of them have to use the formula $\lim\limits_{h\to 0} \ldots$ (or the other formula $\lim\limits_{x\to a} \ldots$).

You can use $a = -1$ right away; or you can compute the formula using a and then plug in -1 at the end.

Method 1. Plug in -1 first.

We get

$$\begin{aligned} g'(-1) &= \lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h} & &= \lim_{h\to 0} \frac{2+h-2}{h(h-2)} \\ &= \lim_{h\to 0} \frac{\frac{2}{(-1+h)-1} - \frac{2}{-1-1}}{h} & &= \lim_{h\to 0} \frac{h}{h(h-2)} \\ &= \lim_{h\to 0} \frac{\frac{2}{h-2} + 1}{h} & &= \lim_{h\to 0} \frac{1}{h-2} \\ &= \lim_{h\to 0} \frac{\frac{2+(h-2)}{h-2}}{h} & &= \frac{1}{0-2} = \boxed{-\frac{1}{2}} \end{aligned}$$

Method 2. Get the formula, then plug in -1 at the end.

We get

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} & &= \lim_{h\to 0} \frac{-2h}{h(a+h-1)(a-1)} \\ &= \lim_{h\to 0} \frac{\frac{2}{(a+h)-1} - \frac{2}{a-1}}{h} & &= \lim_{h\to 0} \frac{-2}{(a+h-1)(a-1)} \\ &= \lim_{h\to 0} \frac{\frac{2(a-1)-2(a+h-1)}{(a+h-1)(a-1)}}{h} & &= \frac{-2}{(a-1)(a-1)} \\ &= \lim_{h\to 0} \frac{2a-2-2a-2h+2}{h(a+h-1)(a-1)} & &= \frac{-2}{(a-1)^2}. \end{aligned}$$

Then plugging in $a = -1$ we get $g'(-1) = \dfrac{-2}{(-1-1)^2} = \dfrac{-2}{4} = \boxed{-\dfrac{1}{2}}$

Notice that we got the same answer both ways.

3. Find the value(s) of x at which the tangent line to the graph of $f(x) = 3x^2e^x$ is horizontal.

The tangent line is horizontal at those places where the derivative is zero. So we take $f'(x)$ and set it equal to 0. Since $f(x)$ is a product, we must use the product rule. We have

$$\begin{gathered} f'(x) = 3x^2e^x + e^x(6x) \overset{\text{set}}{=} 0 \\ 3xe^x(x+2) = 0 \\ 3x = 0 \text{ or } e^x = 0 \text{ or } x + 2 = 0. \end{gathered}$$

Notice that e^x can never be 0. So the only solutions are $\boxed{x = 0 \text{ or } x = -2}$

4. If $g(x) = \dfrac{3x-5}{e^x}$, find $g'(2)$.

Using the quotient rule we have

$$g'(x) = \frac{e^x(3) - (3x-5)e^x}{(e^x)^2} = \frac{(3 - 3x + 5)e^x}{(e^x)^2} = \frac{8-3x}{e^x}.$$

Now to get $g'(2)$ we just plug in $x = 2$:

$$g'(2) = \frac{8 - 3(2)}{e^2} = \boxed{\frac{2}{e^2}}$$

You don't have to simplify your answer to get full credit, but it is good practice!

5. Find the <u>equation</u> of the tangent line to the graph of $h(x) = \sqrt{5x} + 1$ at $x = 4$.

To get the equation of a tangent line, first find the slope. Since we can rewrite $h(x)$ as $\sqrt{5}x^{1/2} + 1$, we can use the power rule to get $h'(x) = \frac{1}{2} \cdot \sqrt{5}x^{-1/2} = \frac{\sqrt{5}}{2\sqrt{x}}$. Therefore the slope at $x = 4$ is $h'(4) = \frac{\sqrt{5}}{2\sqrt{4}} = \frac{\sqrt{5}}{4}$.

Next we plug in $x = 4$ to the function to get the point of tangency (a point on the graph and also on the tangent line). $h(4) = \sqrt{20} + 1 = 2\sqrt{5} + 1$. So the point of tangency is $(4, 2\sqrt{5} + 1)$.

Now we can use the slope and the point of tangency to get the equation. We get

$$\begin{aligned} 2\sqrt{5} + 1 &= \frac{\sqrt{5}}{4} \cdot 4 + b, \qquad \text{so} \\ b &= 2\sqrt{5} + 1 - \frac{\sqrt{5}}{4} \cdot 4 \\ &= 2\sqrt{5} + 1 - \sqrt{5} = \sqrt{5} + 1. \end{aligned}$$

So the equation of the line is $\boxed{y = \frac{\sqrt{5}}{4}x + \sqrt{5} + 1}$.