Math 75B Practice Problems for Midterm II – Solutions
Ch. 16, 17, 12 (E), §§4.7, 4.3, 4.5, 4.4, 3.11 (S)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. The function $f(x) = x^4 - 6x^2$ is increasing on the intervals

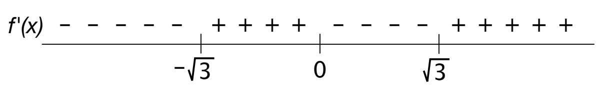
(a) $(0, \sqrt{3})$ only

(b) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ only

(c) $(\sqrt{3}, \infty)$ only

$\boxed{\text{(d)}}$ $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ only

$f'(x) = 4x^3 - 12x \overset{\text{set}}{=} 0$. Solving for x, we get
$4x(x^2 - 3) = 0$
$x = 0 \qquad x = \pm\sqrt{3}$
Now check the number line for $f'(x)$:



$f'(-2) = (-)(+) = (-), \qquad f'(-1) = (-)(-) = (+),$
$f'(1) = (+)(-) = (-), \qquad f'(2) = (+)(+) = (+)$

Therefore the function is increasing on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

2. The function $f(x) = \cos x - x$

(a) is an even function
(b) is an odd function
(c) $\boxed{\text{is neither an even nor an odd function}}$

$f(-x) = \cos(-x) - (-x) = \cos x + x$ (remember that $\cos(-x) = \cos x$!). We have

$$\cos x + x \neq f(x)$$
$$\cos x + x \neq -f(x)$$

Therefore $f(x)$ is neither even nor odd.

3. The function $f(x) = x^4 - 6x^2$ is concave down on the intervals

$\boxed{\text{(a)}}$ $(-1, 1)$ only

(b) $(-\sqrt{3}, \sqrt{3})$ only

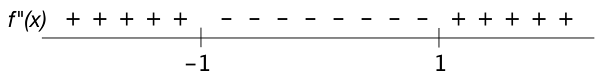
(c) $(-\infty, -1)$ and $(1, \infty)$ only

(d) $(1, \sqrt{3})$ only

To get concavity we have to get the second derivative. So continuing from the previous question, $f''(x) = 12x^2 - 12 \overset{\text{set}}{=} 0$. Solving for x, we get

$$12(x^2 - 1) = 0$$
$$x = \pm 1$$

Now check the number line for $f''(x)$:



$$f''(-2) = (+), \qquad f''(0) = (-), \qquad f''(2) = (+)$$

Therefore the function is concave down on the interval $(-1, 1)$.

4. $\lim_{x \to 1} x^{\frac{1}{\ln x}} =$

(a) 1

(b) 0

$\boxed{\text{(c)}}$ e

(d) does not exist

This is an indeterminate form of type 1^∞, so we use a logarithm to evaluate the limit. We have the related limit $\lim_{x \to 1} \ln \left(x^{\frac{1}{\ln x}} \right)$ which is equal to $\lim_{x \to 1} \frac{1}{\ln x} \ln x$ using a logarithm law. But this is equal to $\lim_{x \to 1} \dfrac{\ln x}{\ln x} = \lim_{x \to 1} 1 = 1$. Therefore the original limit is $e^1 = e$.

5. The linear approximation of $f(x) = \sqrt{5 - x}$ at $x = 1$ is

$\boxed{\text{(a)}}$ $y = -\frac{1}{4}x + \frac{9}{4}$

(b) $y = -\frac{3}{4}x + \frac{7}{4}$

(c) $y = \frac{1}{4}x + \frac{7}{4}$

(d) $y = -\frac{3}{4}x + \frac{9}{4}$

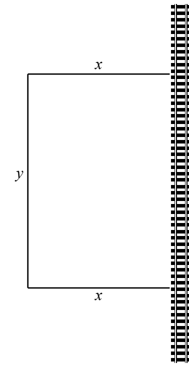
The linear approximation of $f(x)$ at $x = 1$ is the same as the equation of the tangent line at $x = 1$.

$f'(x) = \frac{1}{2}(5 - x)^{-1/2}(-1) = -\frac{1}{2\sqrt{5-x}}$, so $f'(1) = -\frac{1}{2 \cdot \sqrt{5-1}} = -\frac{1}{4}$. The only answer choice with a slope of $-\frac{1}{4}$ is (a).

To make sure that is the right answer, you can also get the y-intercept by plugging in the point $(1, f(1))$ into the equation $y = mx + b$ and solving for b:
$f(1) = \sqrt{5 - 1} = 2$, so $2 = -\frac{1}{4}(1) + b$. We get $b = 2 + \frac{1}{4} = \frac{9}{4}$. Therefore the equation is $y = -\frac{1}{4}x + \frac{9}{4}$, as given.

Fill-In.

1. A contractor has 80 ft. of fencing with which to build three sides of a rectangular enclosure. In order to enclose the largest possible area, the dimensions of the enclosure should be <u>40 ft.</u> × <u>20 ft.</u>



This is a max-min problem. The objective is to **maximize the area**. A formula for the area (see picture) is $A = xy$. To get this formula in terms of a single variable, we need the fact that $2x + y = 80$ (there are only 80 ft. of fencing available). Solving this equation for y, we get $y = 80 - 2x$. So the objective equation becomes

$$A(x) = x(80 - 2x) = 80x - 2x^2.$$

The domain is $0 \le x \le 40$, but the area is 0 at these endpoints. So the area will be maximized at a critical number between 0 and 40.

$A'(x) = 80 - 4x \stackrel{\text{set}}{=} 0 \Rightarrow x = 20$. The area is maximized when $x = 20$. This leaves $y = 40$ ft. for the long side ($20 + 20 + 40 = 80$, or equivalently, using the equation $y = 80 - 2x$ we get $y = 80 - 2(20) = 40$).

2. Using a tangent line approximation, $\sqrt[3]{126.5} \approx$ <u>5.02</u>.

First we get the tangent line approximation to the function $f(x) = \sqrt[3]{x}$ at $x = 125$, since $\sqrt[3]{125} = 5$ is easily computed.

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$, so $f'(125) = \frac{1}{3(125)^{2/3}} = \frac{1}{3\cdot 25} = \frac{1}{75}$. This is the slope of the tangent line. The line passes through the point of tangency $(125, 5)$, so we plug these into $y = mx + b$ and solve for b:

$$\begin{aligned}
5 &= \frac{1}{75}(125) + b \\
5 &= \frac{125}{75} + b \\
5 &= \frac{5}{3} + b \\
b &= 5 - \frac{5}{3} = \frac{15}{3} - \frac{5}{3} = \frac{10}{3}.
\end{aligned}$$

So the equation of the tangent line is $y = \frac{1}{75}x + \frac{10}{3}$. Then $\sqrt[3]{126.5}$ is approximated by plugging in $x = 126.5$ to the tangent line:

$$\sqrt[3]{126.5} \approx \frac{1}{75}(126.5) + \frac{10}{3} = \frac{126.5 + 250}{75} = \frac{376.5}{75}.$$

It is okay to leave your answer in that form, but if you are brave you can convert it to a decimal without a calculator: $\frac{376.5}{75} \cdot \frac{4}{4} = \frac{1506}{300} = \frac{502}{100} = \boxed{5.02}$.

Graphs.

1. For the function $h(x) = \frac{x+5}{x^2-9}$,

   (a) Find the equations of any vertical and/or horizontal asymptotes

   $h(x) = \dfrac{x+5}{x^2-9} = \dfrac{x+5}{(x+3)(x-3)}$, so there are vertical asymptotes at $\boxed{x = 3 \text{ and } x = -3}$

   The domain of $h(x)$ is $x \neq \pm 3$.

   $\lim\limits_{x \to \pm\infty} \dfrac{x+5}{x^2-9} = 0$ since the denominator has larger degree than the numerator. So there is a horizontal asymptote at $\boxed{y = 0}$

   (b) Find the y-intercept

   $h(0) = \frac{5}{-9} = -\frac{5}{9}$, so the y-intercept is the point $\boxed{\left(0, -\frac{5}{9}\right)}$

   (c) Find the x-intercept

   $\frac{x+5}{x^2-9} \overset{\text{set}}{=} 0$ gives $x = -5$ as the only solution. So the x-intercept is the point $\boxed{(-5, 0)}$

   (d) Find the critical **points** and intervals of increase/decrease

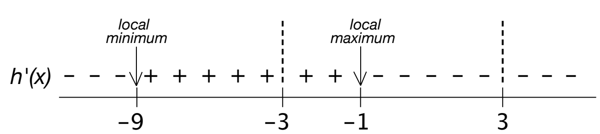
   We have

$$
\begin{aligned}
h'(x) &= \frac{x^2 - 9 - (x+5)\cdot 2x}{(x^2-9)^2} = \frac{x^2 - 9 - 2x^2 - 10x}{(x^2-9)^2} \\
&= \frac{-x^2 - 10x - 9}{(x^2-9)^2} = -\frac{x^2 + 10x + 9}{(x^2-9)^2} \\
&= -\frac{(x+9)(x+1)}{(x^2-9)^2} \overset{\text{set}}{=} 0 \\
&\Rightarrow\ x = -9,\ x = -1.
\end{aligned}
$$

   So the critical numbers are $x = -9$, $x = -1$ ($x = \pm 3$ make $h'(x)$ undefined, but they are not in the domain of $h(x)$ to begin with). Now $h(-9) = -\frac{1}{18}$ and $h(-1) = -\frac{1}{2}$ (check), so the critical points are

   Therefore the critical points are $\boxed{\left(-9, -\frac{1}{18}\right) \text{ and } \left(-1, -\frac{1}{2}\right)}$

   To find the intervals of increase and decrease, we set up a number line for $h'(x)$:

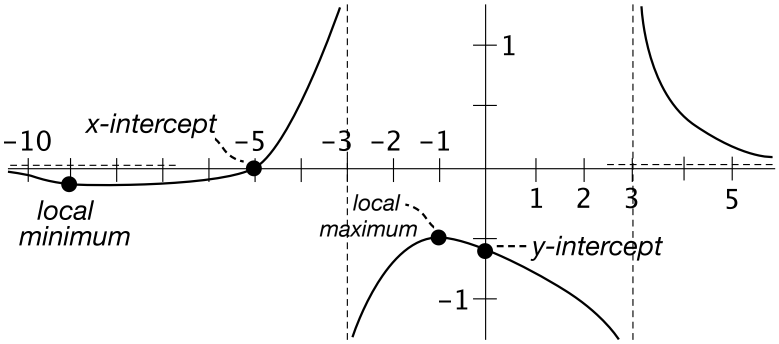


$$
h'(-10) = -\frac{(-)(-)}{(+)} = (-), \qquad h'(-4) = -\frac{(+)(-)}{(+)} = (+),
$$

$$
h'(-2) = -\frac{(+)(-)}{(+)} = (+), \qquad h'(0) = -\frac{(+)(+)}{(+)} = (-), \qquad h'(4) = -\frac{(+)(+)}{(+)} = (-)
$$

   Therefore the function is increasing on the intervals $(-9, -3)$ and $(-3, -1)$, and is decreasing on the intervals $(-\infty, -9)$, $(-1, 3)$ and $(3, \infty)$.

(e) On the axes below, sketch an accurate graph of $h(x)$.

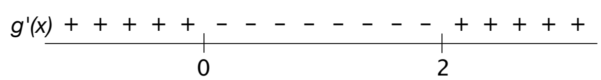


2. For the function $g(x) = \frac{2}{3}x^3 - 2x^2$,

(a) find the critical **points** and intervals of increase/decrease

$g'(x) = 2x^2 - 4x \overset{\text{set}}{=} 0$. Solving for x, we get $2x(x-2) = 0 \Rightarrow x = 0$, $x = 2$. These are the critical **numbers**. $g(0) = 0$ and $g(2) = \frac{2}{3} \cdot 2^3 - 2 \cdot 2^2 = \frac{16}{3} - 8 = \frac{16-24}{3} = -\frac{8}{3}$, so the critical **points** are $(0, 0)$ and $\left(2, -\frac{8}{3}\right)$. Now check the number line for $g'(x)$:

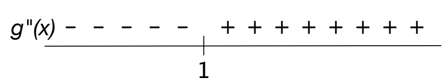
$$g'(-1) = (-)(-) = (+), \qquad g'(1) = (+)(-) = (-), \qquad g'(3) = (+)(+) = (+)$$



Therefore the function is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and decreasing on the interval $(0, 2)$.

(b) find the inflection **points** and intervals of concave up/concave down

To get concavity we have to get the second derivative. $g''(x) = 4x - 4 = 4(x-1) \overset{\text{set}}{=} 0 \Rightarrow$ $x = 1$. So there is an inflection point at $x = 1$. $g(1) = \frac{2}{3} - 2 = \frac{2-6}{3} = -\frac{4}{3}$, so the inflection **point** is $\left(1, -\frac{4}{3}\right)$.
Now check the number line for $g''(x)$: $g''(0) = (-)$, $\qquad g''(2) = (+)$, so we have



Therefore the function is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

(c) discuss any symmetry $g(x)$ may or may not have

$g(x)$ has no symmetry, since (in particular) there is an inflection point at $x = 1$ but not at $x = -1$. You can also check that $g(-x) = \frac{2}{3}(-x)^3 - 2(-x)^2 = -\frac{2}{3}x^3 - 2x^2 \neq g(x)$ and $\neq -g(x)$.

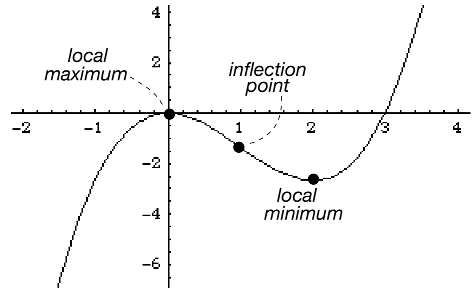
(d) find the equations of any vertical and/or horizontal asymptotes

There are no vertical or horizontal asymptotes because $g(x)$ is a polynomial function.

(e) find the y-intercept

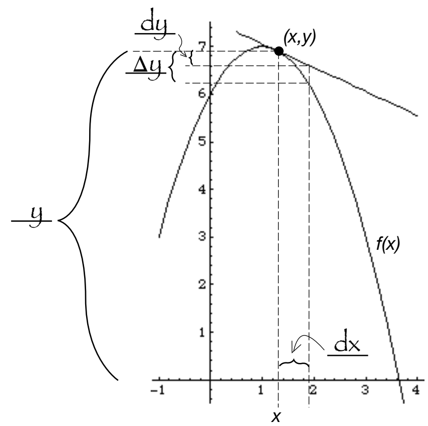
$g(0) = 0$, so the y-intercept is $(0, 0)$.

(f) On the axes at right, sketch an accurate graph of $g(x)$.



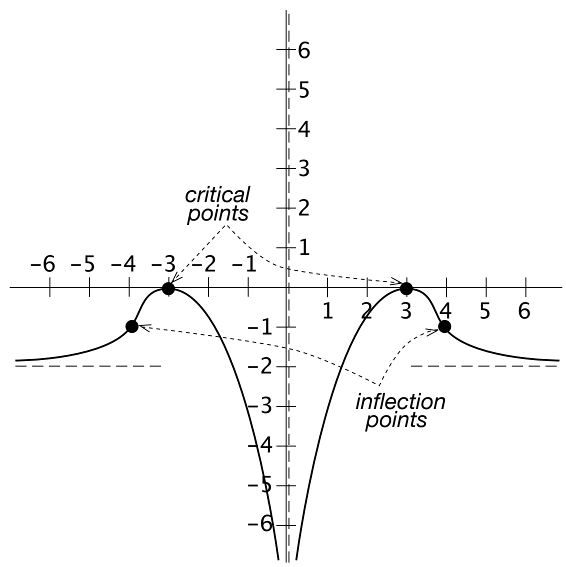
3. The graph of a function $f(x)$ is shown, along with the linear approximation at a point (x, y). Fill in each blank with the correct label from the following list: dx, y, dy, and Δy.

The blanks are filled in at right. Notice that Δy is the *actual* change in the y-value measured on the actual graph of the function — in other words, $\Delta y = f(x+\Delta x) - f(x)$ — whereas dy is the *approximate* change measured on the tangent line.



4. On the axes at right, sketch a graph of a function $f(x)$ satisfying all of the following properties:

- $f(x)$ is an even function
- $(3, 0)$ is a critical point of $f(x)$
- $(4, -1)$ is an inflection point of $f(x)$
- $f(x)$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = -2$
- $f(x)$ is increasing on the intervals $(-\infty, -3)$ and $(0, 3)$
- $f(x)$ is concave up on the intervals $(-\infty, -4)$ and $(4, \infty)$



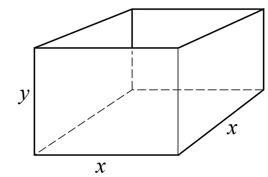
Notice that since $f(x)$ is an even function, it is symmetric about the y-axis. Therefore, since there is a critical point at $(3, 0)$, there must also be a critical point at $(-3, 0)$. Similarly, since there is an inflection point at $(4, -1)$, there must also be an inflection point at $(-4, -1)$. The restrictions above leave very little room for variety in this graph.

Work and Answer. *You must show all relevant work to receive full credit.*

1. If 1200 cm^2 of sheet metal is available to make a box with a square base and open top, find the largest possible volume of the box.

   This is similar to a question on a recent quiz. Now we get to do the problem all the way through!

   The objective of the problem is to **maximize the volume**. A formula for the volume (see picture) is $V = x^2 y$. To get this formula in terms of a single variable, we need the fact that the surface area is $x^2 + 4xy = 1200$. Solving this equation for y, we get



$$4xy = 1200 - x^2$$
$$y = \frac{1200 - x^2}{4x} = 300x^{-1} - \frac{1}{4}x.$$

   So the objective equation becomes

$$V(x) = x^2\left(300x^{-1} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3.$$

   This was exactly the formula I gave you on the blue quiz (the green quiz was a little different since the amount of sheet metal was given to be 2400 cm^2).

   The domain is $0 \le x \le \sqrt{1200}$, but the volume is 0 at these endpoints. So the volume will be maximized at a critical number between 0 and $\sqrt{1200}$.

   $V'(x) = 300 - \frac{3}{4}x^2 \overset{\text{set}}{=} 0 \Rightarrow x^2 = 400 \Rightarrow x = 20$. In other words, the volume is maximized when $x = 20$. Therefore the maximum volume possible is $V = 300(20) - \frac{1}{4} \cdot (20)^3 = 6000 - \frac{8000}{4} = 6000 - 2000 = \boxed{4000\,\text{cm}^3}$.

   Notice! that on the quiz the problem asked for the base length that would maximize the volume, whereas this problem asked for the actual maximum volume. Be sure to pay careful attention to the wording of problems like this so you know what is being asked for.

2. Find the limit $\lim\limits_{t\to 0^+} \ln t \sin t$.

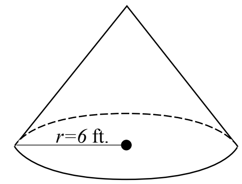
   First recall: we proved in class that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \tag{1}$$

   (you can also verify this using l'Hôpital's Rule). This fact will be useful in this problem. The limit $\lim\limits_{t\to 0^+} \ln t \sin t$ is an indeterminate form of type $0 \cdot (-\infty)$. We have

$$\begin{aligned}
\lim_{t\to 0^+} \ln t \sin t &= \lim_{t\to 0^+} \frac{\ln t}{\csc t} \\
&\overset{\text{H}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\csc t \cot t} \\
&= \lim_{t\to 0^+} -\frac{\sin t \tan t}{t} \\
&= \lim_{t\to 0^+} \frac{\sin t}{t} \cdot \lim_{t\to 0^+} (-\tan t) \qquad \text{(using properties of limits)} \\
&= 1 \cdot 0 \qquad \text{(by (1) above)} \\
&= \boxed{0}
\end{aligned}$$

3. A cone-shaped roof with base radius $r = 6$ ft. is to be covered with a 0.5-inch layer of tar. Use differentials to estimate the amount of tar required (you may use the formula $V(r) = \frac{2}{9}\pi r^3$ for the volume of the piece of the house covered by the roof).



The amount of tar required is approximately $dV = V'(6)dr$, where $dr = \frac{1}{24}$ ft. is the thickness of the tar converted to feet.

[If you need it, on the test I will give you unit conversion formulas such as 12 in. = 1 ft. I will expect you to be able to use this information to calculate conversions such as 0.5 *in.* $= \frac{1}{24}$ *ft. See me before the test if you are unsure how to do this.]*

$V'(r) = \frac{2}{3}\pi r^2$, so $V'(6) = \frac{2}{3}\pi \cdot 6^2 = 24\pi$. Therefore $dV = 24\pi \cdot \frac{1}{24} = \boxed{\pi \text{ ft.}^3}$.