Math 111 Practice Final — Solutions
Ch. 0-9

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

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1. Prove or disprove the following statement:

   Let A, B, and C be sets. Then $(A \cup B) - C = (A - C) \cup (B - C)$.

   The statement is true.

   First we will prove that $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

   Let $x \in (A \cup B) - C$. Then $x \in A \cup B$ and $x \notin C$. Then $x \in A$ or $x \in B$, and $x \notin C$.

   Case 1. $x \in A$.
   Since $x \notin C$, $x \in A - C$. Therefore $x \in (A - C) \cup (B - C)$.

   Case 2. $x \in B$.
   Since $x \notin C$, $x \in B - C$. Therefore $x \in (A - C) \cup (B - C)$.

   Next we will prove that $(A - C) \cup (B - C) \subseteq (A \cup B) - C$.

   Let $x \in (A - C) \cup (B - C)$. Then $x \in A - C$ or $x \in B - C$.

   Case 1. $x \in A - C$.
   Then $x \in A$ and $x \notin C$. Then $x \in A \cup B$ and $x \notin C$, so $x \in (A \cup B) - C$.

   Case 2. $x \in B - C$.
   Then $x \in B$ and $x \notin C$. Then $x \in A \cup B$ and $x \notin C$, so $x \in (A \cup B) - C$.

2. Determine whether the compound propositions $(P \vee Q) \Rightarrow (P \wedge Q)$ and $P \Leftrightarrow Q$ are logically equivalent.

   We construct the truth table for these two propositions:

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \Rightarrow (P \wedge Q)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

   Since for all truth values of P and Q the propositions $(P \vee Q) \Rightarrow (P \wedge Q)$ and $P \Leftrightarrow Q$ have the same truth value, they are logically equivalent.

3. Let $n \in \mathbb{Z}$. Prove that if $3n^2 + 4n + 2$ is even, then n is even.

   We will prove this statement by contrapositive. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} 3n^2 + 4n + 2 &= 3(2k+1)^2 + 4(2k+1) + 2 \\ &= 3(4k^2 + 4k + 1) + 8k + 4 + 2 \\ &= 12k^2 + 20k + 9 \\ &= 2(6k^2 + 10k + 4) + 1. \end{aligned}$$

   Since $6k^2 + 10k + 4 \in \mathbb{Z}$, $3n^2 + 4n + 2$ is odd.

4. Prove or disprove the following statement:

   For any $a \in \mathbb{Z}$, the number $a^3 + a + 100$ is positive.

   The statement is false. For $a = -5$, $a^3 + a + 100 = -125 - 5 + 100 = -30$ is not positive.

5. Consider the relation R defined on $\mathbb{Z}$ by aRb iff $ab \leq 0$. Determine whether R is

   (a) reflexive
   (b) symmetric
   (c) transitive
   (d) an equivalence relation.

   (a) R is not reflexive: e.g. $1 \cdot 1 \nleq 0$, so $(1,1) \notin R$.
   (b) R is symmetric: Suppose $(a,b) \in R$. Then $ab \leq 0$. We have $ba = ab \leq 0$, so $(b,a) \in R$.
   (c) R is not transitive: For a counterexample, $1 \cdot 0 \leq 0$ and $0 \cdot 1 \leq 0$, but $1 \cdot 1 \nleq 0$, so $(1,0) \in R$, $(0,1) \in R$, but $(1,1) \notin R$.
   (d) R is not an equivalence relation since it is not reflexive or transitive.

6. Consider the function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

   Determine whether f is

   (a) one-to-one
   (b) onto
   (c) bijective.

   (a) The function f is not injective: e.g. $f(1) = 2 = f(2)$, but $1 \neq 2$.

(b) We claim that f is not surjective because $1 \notin \text{im}(f)$. To show that 1 is, in fact, a counterexample we will prove the following lemma:

Lemma. For any $x \in \mathbb{Z}$, $f(x)$ is even.

Proof.

Case 1. x is even. Then $f(x) = x$ is even.

Case 2. x is odd. Then $f(x) = 2x$ is even (since $x \in \mathbb{Z}$). □

Since 1 is not even, $f(x) \neq 1$ for any $x \in \mathbb{Z}$, so 1 is not in the image.

Thus the function f is not surjective.

(c) The function is not bijective since it is not injective or surjective.

7. Prove that the number 111 cannot be written as the sum of four integers, two of which are even and two of which are odd.

We will prove this statement by contradiction. Suppose the number 111 *can* be written as the sum of four integers, two of which are even and two of which are odd. Let $111 = a+b+c+d$ where a and b are even and c and d are odd. Then $a = 2k$, $b = 2l$, $c = 2m+1$, and $d = 2n+1$ for some $k, l, m, n \in \mathbb{Z}$. Then $111 = 2k+2l+2m+1+2n+1 = 2(k+l+m+n+1)$. Since $k+l+m+n+1 \in \mathbb{Z}$, 111 is even. Contradiction.

8. Prove that $7 \mid (3^{2n} - 2^n)$ for every nonnegative integer n.

Base Case. ($n = 0$) We have $3^{2n} - 2^n = 3^0 - 2^0 = 0$ which is divisible by 7. Therefore the statement holds for $n = 0$.

Inductive Step. Suppose k is a nonnegative integer for which $7 \mid (3^{2k} - 2^k)$. We must show that $7 \mid (3^{2(k+1)} - 2^{k+1})$. By the inductive hypothesis we have that $3^{2k} - 2^k = 7m$ for some integer m. Therefore $3^{2k} = 7m + 2^k$, and we have

$$3^{2(k+1)} - 2^{k+1} = 9 \cdot 3^{2k} - 2 \cdot 2^k$$
$$= 9(2^k + 7m) - 2 \cdot 2^k = 7 \cdot 9m + 2^k(9-2) = 7(9m + 2^k).$$

Since $9m + 2^k$ is an integer, $7 \mid (3^{2(k+1)} - 2^{k+1})$.

Some kind of **BONUS.**

Possible question: Give an example of a bijective function $f\colon \mathbb{Z} \to \mathbb{N}$ and find its inverse.

Let $f\colon \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 2x+1 & \text{if } x \leq 0 \end{cases}$$

We will prove that f is bijective.

Note that the image of every positive integer is even, and the image of every nonpositive integer is odd.

One-to-One. Suppose $f(a) = f(b)$.

Case 1. $f(a)$ is even. Then $a > 0$ and $b > 0$. Therefore $f(a) = 2a$ and $f(b) = 2b$. It follows that $2a = 2b$, so $a = b$.

Case 2. $f(a)$ is odd. Then $a \leq 0$ and $b \leq 0$. Therefore $f(a) = -2a+1$ and $f(b) = -2b+1$. It follows that $-2a + 1 = -2b + 1$, so $a = b$.

Thus f is injective.

Onto. Let $b \in \mathbb{N}$. We will show that there exists $a \in \mathbb{Z}$ such that $f(a) = b$.

Case 1. b is even. Let $a = \frac{b}{2}$. Since b is even, $a \in \mathbb{Z}$. Since $b > 0$, $a > 0$, so $f(a) = 2a = b$, so $a = b$.

Case 2. b is odd. Then $b = 2k + 1$ for some $k \in \mathbb{Z}$. Let $a = -k$. Since $b \geq 1$, $2k \geq 0$, so $a \leq 0$. Then $f(a) = -2a + 1 = 2k + 1 = b$. Thus f is surjective.

The inverse of f is

$$f^{-1}(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\[2ex] \dfrac{1-x}{2} & \text{if } x \text{ is odd} \end{cases}$$

To verify that this function is the inverse of f , we check $f^{-1} \circ f(x) = x$. We will consider two cases.

Case 1. $x > 0$. Then $f(x) = 2x$ is even, so $f^{-1}(f(x)) = \frac{2x}{2} = x$.

Case 2. $x \leq 0$. Then $f(x) = -2x+1$ is odd, so $f^{-1}(f(x)) = \frac{1-(-2x+1)}{2} = \frac{1+2x-1}{2} = x$.