Math 111 Practice Midterm II — Solutions
Ch. 3-5

For each problem, prove the statement. Indicate what type of proof (trivial, vacuous, direct, by contrapositive, or by contradiction) you are using.

1. Let x be a real number.

   (a) If $x > -7$, then $-5 - x^2 < 0$.

   For any real number x, $x \geq 0$. Therefore $-x^2 \leq 0$, and $-5 - x^2 \leq -5 + 0 = -5 < 0$. (This is a trivial proof.) □

   (b) If $|x| = 5$, then $x^2 + x + 1 > 20$.

   If $|x| = 5$, then either $x = 5$ or $x = -5$. Thus we can consider the following two cases:

   Case 1. $x = 5$.
   Then $x^2 + x + 1 = 5^2 + 5 + 1 = 31 > 20$.

   Case 2. $x = -5$.
   Then $x^2 + x + 1 = (-5)^2 + (-5) + 1 = 21 > 20$. (This is a direct proof by cases.)
   □

   (c) If $2x > x^2 + x^3$, then $x < 1$.

   We will prove this statement by contrapositive. Suppose $x \geq 1$. Then $x^2 \geq x$ and $x^3 \geq x$. Adding these two inequalities gives $x^2 + x^3 \geq 2x$; thus $2x \not> x^2 + x^3$. □

2. Let n and m be integers.

   (a) If $3n^2 + 5n$ is odd, then $n \geq 10$.

   We will show that for any integer n, the number $3n^2 + 5n$ is even. To do this, we will consider two cases:

   Case 1. n is even.
   Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore $3n^2 + 5n = 3(2k)^2 + 5(2k) = 12k^2 + 10k = 2(6k^2 + 5k)$. Since $6k^2 + 5k \in \mathbb{Z}$, the number $3n^2 + 5n$ is even.

   Case 2. n is odd.
   Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $3n^2 + 5n = 3(2k+1)^2 + 5(2k+1) = 12k^2 + 12k + 3 + 10k + 5 = 12k^2 + 22k + 8 = 2(6k^2 + 11k + 4)$. Since $6k^2 + 11k + 4 \in \mathbb{Z}$, the number $3n^2 + 5niseven.$

   Since $3n^2 + 5n$ is never odd, the implication follows. (This is a vacuous proof.) □

   (b) If n is even, then $3n^2 - 2n - 5$ is odd.

Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} 3n^2 - 2n - 5 &= 3(2k)^2 - 2(2k) - 5 \\ &= 12k^2 - 4k - 6 - 1 \\ &= 2(6k^2 - 2k - 3) + 1. \end{aligned}$$

Since $6k^2 - 2k - 3 \in \mathbb{Z}$, the number $3n^2 - 2n - 5$ is odd. (This is a direct proof.) □

(c) If $7n^2 + 4$ is even, then n is even.

We will prove this statement by contrapositive. Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore $7n^2 + 4 = 7(2k + 1)^2 + 4 = 7(4k^2 + 4k + 1) + 4 = 28k^2 + 28k + 11 = 2(14k^2 + 14k + 5) + 1$. Since $14k^2 + 14k + 5 \in \mathbb{Z}$, $7n^2 + 4$ is odd. □

(d) If $n - 5m$ is odd, then n and m are of opposite parity.

We will prove the statement by contrapositive. Thus we will prove that if n and m are of the same parity, then $n - 5m$ is even.

Case 1. n and m are both even.
Then $n = 2k$ and $m = 2l$ for some $k, l \in \mathbb{Z}$. Therefore $n - 5m = 2k - 5(2l) = 2k - 10l = 2(k - 5l)$. Since $k - 5l \in \mathbb{Z}$, the number $n - 5m$ is even.

Case 1. n and m are both odd.
Then $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$. Then $n - 5m = 2k + 1 - 5(2l + 1) = 2k + 1 - 10l - 5 = 2k - 10l - 4 = 2(k - 5l - 2)$. Since $k - 5l - 2 \in \mathbb{Z}$, the number $n - 5m$ is even.

(e) If $5 \mid (n - 1)$, then $5 \mid (n^3 + n - 2)$.

Suppose $5 \mid (n-1)$. Then $n \equiv 1 \pmod 5$. Therefore $n^3 + n - 2 ?1^3 + 1 - 2 \equiv 0 \pmod 5$. This implies that $5 \mid (n^3 + n - 2)$. (This is a direct proof.)

Another proof: Suppose $5 \mid (n - 1)$. Then $n - 1 = 5k$ for some $k \in \mathbb{Z}$. Therefore $n = 5k + 1$, and we have $n^3 + n - 2 = (5k + 1)^3 + (5k + 1) - 2 = 125k^3 + 75k^2 + 15k + 1 + 5k + 1 - 2 = 125k^3 + 75k^2 + 20k = 5(25k^3 + 15k^2 + 4k)$. Since $25k^3 + 15k^2 + 4k \in \mathbb{Z}$, we have $5 \mid n^3 + n - 2$. (This is also a direct proof.) □

(f) $3 \mid mn$ if and only if $3 \mid m$ or $3 \mid n$.

We have two implications to prove.

(⇒) Suppose $3 \mid mn$. Show that $3 \mid m$ or $3 \mid n$.
We will prove this statement by contrapositive. Suppose $3 \nmid m$ **and** $3 \nmid n$. Then $m = 3k + c$ and $n = 3l + d$ for some $k, l, c, d \in \mathbb{Z}$ where c and d are equal to either 1 or 2. We have

$$\begin{aligned} mn &= (3k + c)(3l + d) \\ &= 9kl + 3kd + 3lc + cd \\ &= 3(3kl + kd + lc) + cd. \end{aligned}$$

Case 1. $c = d = 1$.
Then $cd = 1$, and $mn = 3(3kl + k + l) + 1$. Thus $3 \nmid mn$.

Case 2. WLOG $c = 2$, $d = 1$.
Then $cd = 2$, and $mn = 3(3kl + 2k + l) + 2$. Thus $3 \nmid mn$.

Case 3. $c = d = 2$.
Then $cd = 4$, and $mn = 3(3kl + 2k + 2l) + 4 = 3(3kl + 2k + 2l + 1) + 1$. Thus $3 \nmid mn$.

($\Leftarrow$) Suppose $3 \mid m$ or $3 \mid n$. Show that $3 \mid mn$.
We will prove this statement directly. Without loss of generality, suppose $3 \mid m$. Then $m = 3k$ for some $k \in \mathbb{Z}$. Therefore $mn = 3kn$ is divisible by 3. □

3. The number $\log_3 2$ is irrational.

Suppose $\log_3 2$ is rational. Then $\log_3 2 = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n > 0$. Therefore $3^{m/n} = 2$, so $3^m = 2^n$. Since $n > 0$, $3^m = 2^n > 1$. We know that 3^m is odd since it is a product of odd integers (or you can prove a little lemma here that says, *If a is odd and m is a positive integer, then a^m is odd.*). But we also know that 2^n is even since it is a product of even integers (or you can prove another similar lemma if you are not convinced). This is a contradiction, since an odd number cannot be equal to an even number. Therefore $\log_3 2$ is irrational. (This is a proof by contradiction.) □

4. The product of a nonzero rational number and an irrational number is irrational.

Suppose there exist a nonzero rational number x and an irrational number y such that xy is rational. Then $x = \frac{k}{l}$ for some $k, l \in \mathbb{Z}$, $k, l \neq 0$ and $xy = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $m \neq 0$. We have $y = \frac{xy}{x} = \frac{\frac{m}{n}}{\frac{k}{l}} = \frac{ml}{nk}$. Since $ml, nk \in \mathbb{Z}$ and $nk \neq 0$, y is rational. Contradiction. (This is a proof by contradiction.) □

5. Let A and B be sets. Then $A \cap B = \emptyset$ if and only if $(A \times B) \cap (B \times A) = \emptyset$.

We have two implications to prove.

($\Rightarrow$) Suppose $A \cap B = \emptyset$. Show that $(A \times B) \cap (B \times A) = \emptyset$.
We will prove this statement by contrapositive. Suppose $(A \times B) \cap (B \times A) \neq \emptyset$. Then there exists an element $x \in (A \times B) \cap (B \times A)$. Therefore $x \in A \times B$ and $x \in B \times A$. Thus $x = (y, z)$ for some $y \in A \cap B$ (and $z \in B \cap A$) It follows that $A \cap B \neq \emptyset$.

($\Leftarrow$) Suppose $(A \times B) \cap (B \times A) = \emptyset$. Show that $A \cap B = \emptyset$.
We will prove this statement by contrapositive as well. Suppose $A \cap B \neq \emptyset$. Then there exists $x \in A \cap B$. Since $x \in A$ and $x \in B$, we have $(x, x) \in A \times B$ and $(x, x) \in B \times A$. Thus $(x, x) \in (A \times B) \cap (B \times A)$ and $(A \times B) \cap (B \times A) \neq \emptyset$. □