Math 111 Practice Midterm III – Solutions
Ch. 6-9

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1. Prove or disprove the following statements.

   (a) There exists a nonzero integer a such that for every real number b, $b^2 \geq a$.

   This statement is true. For example, let $a = -1$. Then for every real number b, we have $b^2 \geq 0 \geq -1$, so $b^2 \geq a$.

   (b) There exists an integer a such that $a^3 + 2a + 3 = 100$.

   This statement is false. For any integer a, either $a \leq 4$ or $a \geq 5$. If $a \leq 4$, then $a^3 + 2a + 3 \leq 4^3 = 2 \cdot 4 + 3 = 75 < 100$. If $a \geq 5$, then $a^3 + 2a + 3 \geq 5^3 + 2 \cdot 5 + 3 = 138 > 100$. Therefore $a^3 + 2a + 3 = 100$ is false for every integer a.

   (c) For any integer a there exists an integer b such that $b^2 = a$.

   This statement is false. For example, if $a = -1$, then there is no integer b such that $b^2 = -1$.

   (d) The sum of any two positive irrational numbers is irrational.

   This statement is false. For example, $\sqrt{2} + (2 - \sqrt{2}) = 2$. We proved in class that $\sqrt{2}$ is irrational. You can make a similar argument to show that $2 - \sqrt{2}$ is also irrational.

   (e) Any irrational number is the sum of an irrational number and a positive rational number.

   This statement is true. Let a be any irrational number. Then $a = 1 + (a - 1)$. Observe that 1 is rational, and you can prove that $a - 1$ is irrational, again similar to arguments in the previous problem.

   (f) For any sets A and B there exists a set C such that $A \cup C = B \cup C$.

   This statement is true. Let $C = A \cup B$. Then $A \cup C = A \cup A \cup B = A \cup B$ and $b \cup C = B \cup A \cup B = A \cup B$, so $A \cup C = B \cup C$.

   (g) Let A, B, C, and D be sets such that $A \subseteq C$ and $B \subseteq D$. If $A \cap B = \emptyset$, then $C \cap D = \emptyset$.

   This statement is false. For example, if $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, $D = \{2, 3\}$, then $A \subseteq C$, $B \subseteq D$, and $A \cap B = \emptyset$, and yet $C \cap D \neq \emptyset$.

   (h) Let A, B, C, and D be sets such that $A \subseteq C$ and $B \subseteq D$. If $C \cap D = \emptyset$, then $A \cap B = \emptyset$.

   This statement is true. Suppose that $A \subseteq C$, $B \subseteq D$, $C \cap D = \emptyset$, but $A \cap B \neq \emptyset$. Then there is an element $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $A \subseteq C$ and $B \subseteq D$, it follows that $x \in C$ and $x \in D$. Then $x \in C \cap D$, thus $C \cap D \neq \emptyset$, a contradication.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Which of the following are relations from A to B or relations from B to A? Which of them are functions?

   (a) $\{(a, 1), (b, 2), (c, 3)\}$

   This is a relation from B to A (since it is a subset of $B \times A$). Moreover, it is a function from B to A since each element of B is the first coordinate of exactly one pair in the relation.

   (b) $\{(1, b), (1, c), (3, a), (4, b)\}$

   This is a relation from A to B (since it is a subset of $A \times B$), but it is not a function since the image of 1 is not well-defined.

3. Determine which of the following relations are reflexive; symmetric; transitive. Which of them are equivalence relations? For those that are, describe the distinct equivalence classes.

   (a) Relation R on set $\mathbb{Z}$ defined by $(a, b) \in R$ iff $a + b = 0$.

   Reflexive. R is not reflexive since $1 + 1 \neq 0$ and thus $1 \not R 1$.

   Symmetric. R is symmetric: Suppose aRb. Then $a + b = 0$. Thus $b + a = 0$, and bRa.

   Transitive. R is not transitive since $-1R1$ and $1R-1$ but $1 \not R 1$.

   R is not an equivalence relation since R is not reflexive or transitive.

   (b) Relation R on set $\mathbb{R}$ defined by $(a, b) \in R$ iff $\frac{a}{b} \in \mathbb{Q}$.

   Reflexive. R is not reflexive since $\frac{0}{0} \notin \mathbb{Q}$ and thus $0 \not R 0$.

   Symmetric. R is not symmetric since $\frac{0}{1} = 0 \in \mathbb{Q}$ but $\frac{1}{0} \notin \mathbb{Q}$, i.e. $0R1$ but $1 \not R 0$.

   Transitive. R is transitive: if aRb and bRc, then $\frac{a}{b}$ and $\frac{b}{c} \in \mathbb{Q}$; thus $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} \in \mathbb{Q}$, and aRc.

   R is not an equivalence relation since R is not reflexive or symmetric.

   (c) Relation R on set $\mathbb{R}$ defined by $(a, b) \in R$ iff $ab > 0$.

   Reflexive. R is not reflexive since $0 \cdot 0 \not> 0$ so $0 \not R 0$.

   Symmetric. R is symmetric: If $ab > 0$, then $ba > 0$.

   Transitive. R is transitive: If $ab > 0$ and $bc > 0$, then either a, b, and c are all positive or they are all negative. In either case, $ac > 0$.

   R is not an equivalence relation since R is not reflexive.

   (d) Relation R on set $\mathbb{Z}$ defined by $(a, b) \in R$ iff $a \equiv b \pmod 3$.

Reflexive. R is reflexive: for any $a \in \mathbb{Z}$, we have $a \equiv a \pmod 3$, so aRa.
Symmetric. R is symmetric: if $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$.
Transitive. R is transitive: if $a \equiv b \pmod 3$ and $b \equiv c \pmod 3$, then $a \equiv c \pmod 3$.

R is an equivalence relation since it is reflexive, symmetric and transitive.
The equivalence classes are

$$\begin{aligned}
[0] &= \{a \in \mathbb{Z} \mid a \equiv 0 \pmod 3\} = \{\dots, -3, 0, 3, 6, \dots\} \\
[1] &= \{a \in \mathbb{Z} \mid a \equiv 1 \pmod 3\} = \{\dots, -2, 1, 4, 7, \dots\} \\
[2] &= \{a \in \mathbb{Z} \mid a \equiv 2 \pmod 3\} = \{\dots, -1, 2, 5, 8, \dots\}
\end{aligned}$$

(e) Relation R on set $\mathbb{Q}$ defined by $(a, b) \in R$ iff $a > b$.

Reflexive. R is not reflexive since $1 \not> 1$ so $1 \not R 1$.
Symmetric. R is not symmetric since $2 > 1$ but $1 \not> 2$.
Transitive. R is transitive: Suppose $a > b$ and $b > c$. Then $a > c$.

R is not an equivalence relation since R is not reflexive or symmetric.

4. Determine which of the following functions are one-to-one; onto; bijective.

(a) $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 5n^2 + 2$.

One-to-One. f is not one-to-one since $f(1) = f(-1) = 2$.
Onto. f is not onto since $3 \notin \operatorname{im}(f)$ (the only real solutions to the equation $5n^2 + 2 = 3$ are $\pm\frac{1}{\sqrt{5}}$ which are not integers).

f is not bijective since f is neither one-to-one nor onto.

(b) $f\colon \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \frac{1}{n}$.

One-to-One. f is one-to-one: if $f(x) = f(y)$, then $\frac{1}{x} = \frac{1}{y}$ and by cross-multiplying we get $x = y$.
Onto. f is not onto since $2 \notin \operatorname{im}(f)$ (the only real solution to the equation $\frac{1}{n} = 2$ is $n = \frac{1}{2}$ which is not a natural number).

f is not bijective since f is not onto.

(c) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

One-to-One. f is one-to-one: suppose $f(x) = f(y)$. If $f(x) = f(y) = 0$, then $x = y = 0$. If not then $\frac{1}{x} = \frac{1}{y}$, and similar to the previous problem we get $x = y$.

Onto. f is onto: Suppose $y \in \mathbb{R}$. If $y = 0$ then $f(0) = 0 = y$. If $y \neq 0$ then $f\left(\frac{1}{y}\right) = y$. Therefore $y \in \text{im}(f)$.

f is bijective since f is one-to-one and onto.

(d) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$.

One-to-One. f is not one-to-one since $f(1) = f(0) = 0$.
Onto. f is onto since it is a continuous function whose end behavior is $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

f is not bijective since f is not one-to-one.

5. Prove or disprove the following statements.

(a) Let $f\colon A \to B$ and $g\colon B \to C$ be two functions. If g is onto, then $g \circ f$ is onto.

This statement is false. Consider the following example: $A = B = C = \{1, 2\}$, $f(1) = f(2) = 1$, $g(1) = 1$, $g(2) = 2$. Then $g \circ f(1) = g \circ f(2) = 1$. Note that g is onto, but $g \circ f$ is not.

(b) Let $f\colon A \to B$ and $g\colon B \to C$ be two functions. If both g and $g \circ f$ are one-to-one, then f is one-to-one.

This statement is true. Suppose $f(x) = f(y)$ for some x, $y \in A$. then $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$. Since $g \circ f$ is one-to-one, $x = y$. Hence f is one-to-one.

Note: we did not use the fact that g is one-to-one.

(c) Let $f\colon A \to B$ and $g\colon B \to C$ be two functions. If both f and $g \circ f$ are one-to-one, then g is one-to-one.

The statement is false. Consider the following example: $A = C = \{1\}$, $B = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. Then $g \circ f(1) = 1$. Note that f and $g \circ f$ are one-to-one, but g is not.

6. Use mathematical induction to prove the following statements.

(a) Let $n \in \mathbb{N}$. Then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

Base Case. Let $n = 1$. Then $\dfrac{n(n+1)(n+2)}{3} = \dfrac{1(1+1)(1+2)}{3} = 2 = 1 \cdot 2$. Thus the statement holds for $n = 1$.
Inductive Step. Suppose k is a positive integer for which $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$. We must show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k+1) + (k+1)(k+2) = \dfrac{(k+1)(k+2)(k+3)}{3}$.

We have

$$\begin{aligned}
1\cdot 2+2\cdot 3+3\cdot 4+...+k(k+1)+(k+1)(k+2) &= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \text{(by in}\\
&= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}\\
&= \frac{(k+1)(k+2)(k+3)}{3}.
\end{aligned}$$

(b) Let $n \in \mathbb{N}$. Then $5 \mid (n^5 - n)$.

Base Case. Let $n = 1$. Then $n^5 - n = 1^5 - 1 = 0$ which is divisible by 5.

Inductive Step. Suppose k is a positive integer for which $5 \mid (k^5 - k)$. We must show that $5 \mid ((k+1)^5 - (k+1))$.
We know by the inductive hypothesis that $k^5 - k = 5m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned}
((k+1)^5 - (k+1)) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1\\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k\\
&= 5m + 5k^4 + 10k^3 + 10k^2 + 5k\\
&= 5(m + k^4 + 2k^3 + 2k^2 + k)
\end{aligned}$$

which is divisible by 5.