Math 75A Practice Midterm III Solutions
Ch. 6-8 (Ebersole), §§2.7-3.4 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $g(x) = 3x^4 \sin x$, then $g'(x) = 12x^3 \cos x$. **T** $\boxed{\textbf{F}}$

   The above statement is the "false product rule"! The correct product rule gives $g'(x) = 3x^4 \cos x + 12x^3 \sin x$.

2. $\sec\theta \tan\theta = \dfrac{\sin\theta}{\cos^2\theta}$ for all angles θ. $\boxed{\textbf{T}}$ **F**

   Notice that $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. So $\sec\theta\tan\theta = \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\cos^2\theta}$.

3. $\sin(5t) = 5\sin t$ for all angles t. **T** $\boxed{\textbf{F}}$

   This is almost *never* true! For example, if you plug in $t = \dfrac{\pi}{2}$, on the left hand side you get $\sin\left(\dfrac{5\pi}{2}\right) = 1$, whereas on the right hand side you get $5\sin\left(\dfrac{\pi}{2}\right) = 5 \cdot 1 = 5$. So they are not the same!

4. $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$. $\boxed{\textbf{T}}$ **F**

   The reference angle for $\dfrac{2\pi}{3}$ (the angle in quadrant I corresponding to $\dfrac{2\pi}{3}$) is $\dfrac{\pi}{3}$. Using our special triangle we get $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$. Since $\dfrac{2\pi}{3}$ is in quadrant II, and the tangent function is always negative in quadrant II, we get $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$.

5. The only solution to the equation $\cos t = -1$ is $t = \pi$. **T** $\boxed{\textbf{F}}$

   Any angle that is coterminal with π also satisfies the equation. For example, $t = -\pi$, $t = \pm 3\pi$, $t = \pm 5\pi$, etc.

Multiple Choice. *Circle the letter of the best answer.*

1. The inverse of the function $f(x) = 3x^4 + 1$ is

   (a) $f^{-1}(x) = \sqrt[4]{\dfrac{x-1}{3}}$

   (b) $f^{-1}(x) = \sqrt[4]{\dfrac{x}{3}} - 1$

   (c) $f^{-1}(x) = \sqrt[4]{\dfrac{x}{3} - 1}$

   (d) $\boxed{\text{none of these; } f(x) \text{ does not have an inverse}}$

$f(x)$ is a polynomial of degree 4; therefore it is not one-to-one and does not have an inverse.

2. $\dfrac{8^t 16^3}{2^t} =$

(a) 2^{3t-12}

(b) $\boxed{2^{2t+12}}$

(c) 2^{12-t}

(d) 2^{8t+3}

We have

$$\begin{aligned} \frac{8^t 16^3}{2^t} &= \frac{(2^3)^t (2^4)^3}{2^t} \\ &= \frac{2^{3t} 2^1 2}{2^t} \\ &= \frac{2^{3t+12}}{2^t} \\ &= 2^{3t+12-t} = 2^{2t+12}. \end{aligned}$$

3. The inverse of the function $f(x) = 5x^3$ is $f^{-1}(x) =$

(a) $\dfrac{\sqrt[3]{x}}{5}$

(b) $5\sqrt[3]{x}$

(c) $\boxed{\sqrt[3]{\dfrac{x}{5}}}$

(d) $\dfrac{1}{5x^3}$

To find the inverse of $f(x)$ we "undo" what has been done to x to get x by itself. Then we switch x and y (or you can switch x and y first and then solve for the new y, whichever you prefer). We have

$$\begin{aligned} y &= 5x^3 \\ \frac{y}{5} &= x^3 \\ x &= \sqrt[3]{\frac{y}{5}} \\ y = f^{-1}(x) &= \sqrt[3]{\frac{x}{5}}. \end{aligned}$$

4. If $\ln(3x-2) = 4$, then $x =$

(a) $\boxed{\dfrac{e^4+2}{3}}$

(b) $\dfrac{\ln 4 + 2}{3}$

(c) $\dfrac{4 + \ln 2}{\ln 3}$

(d) $e^{4/3} + 2$

$\ln(3x-2) = 4$ means $e^4 = 3x - 2$. Solving for x we get

$$\begin{aligned} 3x &= e^4 + 2 \\ x &= \frac{e^4+2}{3}. \end{aligned}$$

5. If $3e^{2x-5} = 4^x$, then $x =$

(a) $\dfrac{3-\ln 5}{2-\ln 4}$

(b) $\dfrac{\ln 15}{\ln 8}$

(c) $\dfrac{5+\ln 3}{4+\ln 2}$

(d) $\boxed{\dfrac{5-\ln 3}{2-\ln 4}}$

The bases on each side are different, so we will need to pick a base and use logarithm laws. In light of the answer choices we should choose base e (i.e. natural log); we have

$$\begin{aligned}
\ln\left(3e^{2x-5}\right) &= \ln\left(4^x\right) \\
\ln 3 + \ln\left(e^{2x-5}\right) &= x\ln(4) \qquad \text{(be careful of the 3 on the left side!)} \\
\ln 3 + 2x - 5 &= (\ln 4)x \\
2x - (\ln 4)x &= 5 - \ln 3 \\
(2-\ln 4)x &= 5 - \ln 3 \\
x &= \frac{5-\ln 3}{2-\ln 4}.
\end{aligned}$$

6. The inverse of the function $f(x) = 7^{3x-2}$ is $f^{-1}(x) =$

(a) $\boxed{\dfrac{\log_7(x)+2}{3}}$

(b) $\log_7\left(\dfrac{3x}{2}\right)$

(c) $3\log_7(x)+2$

(d) $(3x-2)^7$

We proceed as in #3; we have

$$\begin{aligned}
y &= 7^{3x-2} \\
\log_7(y) &= 3x - 2 \qquad \text{(since } \log_7(\clubsuit) \text{ “undoes” } 7^{\clubsuit}\text{)} \\
\log_7(y) + 2 &= 3x \\
x &= \frac{\log_7(y)+2}{3} \\
y = f^{-1}(x) &= \frac{\log_7(x)+2}{3}
\end{aligned}$$

7. The inverse of the function $f(x) = 3\ln(5x-2)$ is $f^{-1}(x) =$

(a) $e^{\frac{5x-2}{3}}$

(b) $\dfrac{1}{3}e^{5x-2}$

(c) $\dfrac{e^{2x-3}}{5}$

(d) $\boxed{\dfrac{e^{x/3}+2}{5}}$

Again we proceed as in #3; we have

$$\begin{aligned} y &= 3\ln(5x-2) \\ \frac{y}{3} &= \ln(5x-2) \\ e^{y/3} &= 5x-2 \qquad \text{(since } e^{\clubsuit} \text{ "undoes" } \ln(\clubsuit)) \\ e^{y/3}+2 &= 5x \\ x &= \frac{e^{y/3}+2}{5} \\ y = f^{-1}(x) &= \frac{e^{x/3}+2}{5} \end{aligned}$$

8. If $f(x) = \tan x$, then $f'(x) =$

   (a) $\boxed{\sec^2 x}$

   (b) $\dfrac{\sin x}{\cos x}$

   (c) $\dfrac{1}{\tan x}$

   (d) $\sec x \tan x$

   This is one of the formulas you should memorize. However, if you forget it you can remember that $\tan x = \dfrac{\sin x}{\cos x}$ and use the quotient rule to get the derivative, as we did in class.

9. If $f(x) = x\tan x$, then $f'(x) =$

   (a) $\sec^2 x$

   (b) $x\sec^2 x$

   (c) $\boxed{x\sec^2 x + \tan x}$

   (d) $2x\sec^2$

   Using the product rule, we have $f'(x) = x\sec^2 x + \tan x$.

10. If $f(x) = \tan(3\sqrt{x} + e^x)$, then $f'(x) =$

    (a) $\sec^2(3\sqrt{x}+e^x)$

    (b) $\sec^2(3\sqrt{x}+e^x)\dfrac{3}{2\sqrt{x}} + e^x$

    (c) $\boxed{\sec^2(3\sqrt{x}+e^x)\left(\dfrac{3}{2\sqrt{x}} + e^x\right)}$

    (d) $\sec^2(x)\left(\dfrac{3}{2\sqrt{x}} + e^x\right)$

    Using the chain rule with $3\sqrt{x}+e^x$ as the "chunk" we have $f'(x)$ as in choice (c).

11. If $f(x) = 4^x$, then $f'(x) =$

    (a) $\boxed{(\ln 4)\cdot 4^x}$

    (b) 4^x

    (c) $x \cdot 4^{x-1}$

    (d) $\ln(4^x)$

Another formula to memorize carefully. Or ... here's a trick for deriving the formula:

Notice that $4^x = e^{\ln(4^x)} = e^{x\ln(4)}$ (using logarithm laws and the fact that $e^{\clubsuit}$ "undoes" $\ln(\clubsuit)$). So the derivative is then e^{chunk}(derivative of chunk) $= e^{x\ln(4)}(\ln(4)) = 4^x \cdot \ln(4)$. This trick can come in handy if you forget the formula!

12. If $f(t) = (\tan t)e^{3t}$, then $f'(t) =$

(a) $3e^{3t}\sec^2 t$

(b) $\boxed{3e^{3t}\tan t + e^{3t}\sec^2 t}$

(c) $\tan^3 t \sec^2(e^t)$

(d) $e^{3t}\tan t + 3e^{3t}\sec^2 t$

Using the product rule (and the chain rule), we have $f'(t) = \tan t \cdot e^{3t} \cdot 3 + e^{3t}\sec^2 t = 3e^{3t}\tan t + e^{3t}\sec^2 t$.

13. If $f(x) = \ln(\sin^7 x)$, then $f'(x) =$

(a) $\boxed{7\cot x}$

(b) $7\ln(\sin x)$

(c) $\dfrac{1}{\sin^7 x}$

(d) $\dfrac{7\sin^6 x\cos x}{x}$

First of all, recall that the derivative of $\ln(x)$ is $\dfrac{1}{x}$, so using the chain rule we have that the derivative of ln(chunk) is $\dfrac{1}{\text{chunk}}$(derivative of chunk).

There are two ways to do this problem. One is to use a logarithm law to rewrite $f(x)$ in a simpler way: $f(x) = 7\ln(\sin x)$. Then the derivative is $f'(x) = 7 \cdot \dfrac{1}{\sin x} \cdot \cos x = 7\cot x$.

If you don't think of that, you can still get the answer using the chain rule; you will just have to do some more simplification at the end. We have $f'(x) = \dfrac{1}{\sin^7(x)} 7\sin^6(x) \cdot \cos x = \dfrac{7\cos x}{\sin x} = 7\cot x$.

14. If $f(x) = \ln((2x-1)^{35}(5x^6+7)^{10})$, then $f'(x) =$

(a) $\boxed{\dfrac{70}{2x-1} + \dfrac{300x^5}{5x^6+7}}$

(b) $\dfrac{1}{(2x-1)^{35}(5x^6+7)^{10}}$

(c) $\dfrac{35}{2x-1} + \dfrac{10}{5x^6+7}$

(d) $\dfrac{1}{(2x-1)^{35}} + \dfrac{1}{(5x^6+7)^{10}}$

Similar to #13, you *can* do this problem in two ways. However, it is *much* easier to use logarithm laws first, so that's what we'll do here. We have $f(x) = 35\ln(2x-1) + 10\ln(5x^6+7)$, so $f'(x) = 35 \cdot \dfrac{1}{2x-1} \cdot 2 + 10 \cdot \dfrac{1}{5x^6+7}(30x^5) = \dfrac{70}{2x-1} + \dfrac{300x^5}{5x^6+7}$.

15. If $f(x) = \dfrac{\ln x}{x^3}$, then $f'(x) =$

(a) $\dfrac{1}{3x^3}$

(b) $3x^2 \cdot \dfrac{1}{x} + x^3 \ln x$

(c) $\boxed{\dfrac{1 - 3\ln x}{x^4}}$

(d) $\dfrac{3x^2}{\ln x}$

Using the quotient rule we have

$$\begin{aligned} f'(x) &= \frac{x^3 \cdot \frac{1}{x} - \ln x \cdot 3x^2}{(x^3)^2} \\ &= \frac{x^2 - 3x^2 \ln x}{x^6} \\ &= \frac{x^2(1 - 3\ln x)}{x^6} = \frac{1 - 3\ln x}{x^4}. \end{aligned}$$

16. If \$1000 is invested at 2% interest compounded annually, the balance after 10 years is

(a) $1000e^{1.02}$

(b) $2000e^{10}$

(c) $1000 \cdot 2^{10}$

(d) $\boxed{1000(1.02)^{10}}$

The formula for interest compounded annually is $B(t) = P(1+r)^t$, where P is the "principal" or amount invested, r is the interest rate as a decimal, and t is the number of years. So we have $B(10) = 1000(1 + 0.02)^{10} = 1000(1.02)^{10}$.

Fill-In. *Fill in the correct integer (positive, negative, or 0 whole number).*

1. $\log_2\left(\dfrac{1}{32}\right) = \underline{\ -5\ }$

   $\dfrac{1}{32} = 2^{-5}.$

2. $\log(100) = \underline{\ 2\ }$

   $100 = 10^2.$

3. $e^{4\ln 10} = \underline{\ 10000\ }$

   $e^{4\ln 10} = e^{\ln 10^4} = 10^4 = 10000.$

4. $\log_4(12) - \log_4(3) = \underline{\ 1\ }$

   Using a law of logarithms, $\log_4(12) - \log_4(3) = \log_4(12/3) = \log_4(4) = 1.$

5. $3^{2\log_9(2007)} = \underline{\ 2007\ }$

   We have $3^{2\log_9(2007)} = 9^{\log_9(2007)} = 2007.$

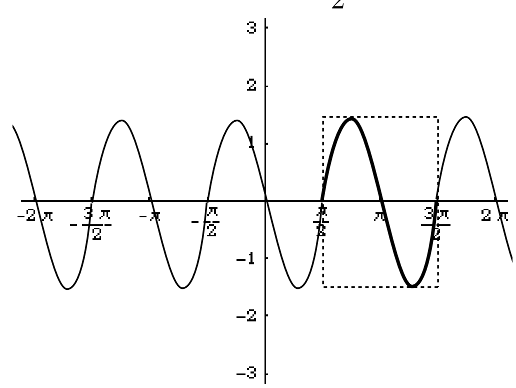
6. $\log_{1/4}(64) = \underline{\ -3\ }$

   $64 = \left(\dfrac{1}{4}\right)^{-3}.$

Graphs. *More accuracy = more points!*

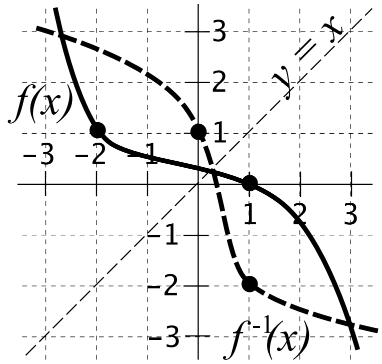
1. On the axes at right, sketch a graph of at least one period of the function $f(t) = \dfrac{3}{2}\sin(2t - \pi)$.

   The first thing to do is to rewrite the function as $f(t) = \frac{3}{2}\sin\left(2\left(t - \frac{\pi}{2}\right)\right)$, so that we can see that the phase shift (horizontal shift) is $\dfrac{\pi}{2}$ to the right. The box (dashed lines) shows where one period of the function goes. Notice that the amplitude is $A = \dfrac{3}{2}$, the period is $\dfrac{2\pi}{2} = \pi$, and there is no vertical shift.



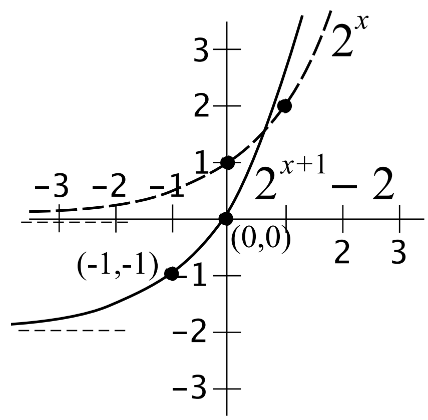
2. For the graph of $f(x)$ shown at right, sketch a graph of $f^{-1}(x)$ on the same axes.

The graph of the inverse of a function $f(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$. You can also plot points by remembering that if (a, b) is on the graph of $f(x)$, then (b, a) is on the graph of $f^{-1}(x)$. For example, the function $f(x)$ in this problem passes through the points $(-2, 1)$ and $(1, 0)$, so the inverse must pass through $(1, -2)$ and $(0, 1)$. Putting these ideas together, we get the graph shown.



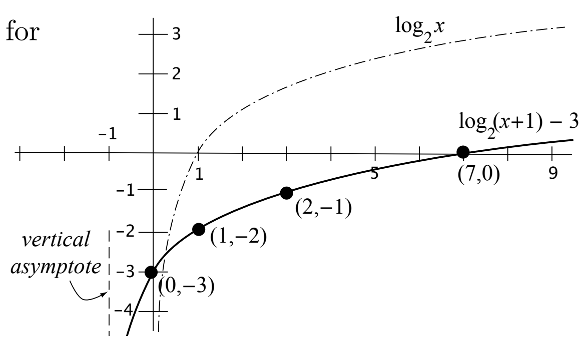
3. On the axes below, sketch a graph of $f(x) = 2^{x+1} - 2$.
Label at least two points on the curve.

The graph of $f(x)$ can be obtained by shifting the graph of $g(x) = 2^x$ to the left 1 unit and down 2 units. The graph at right shows both $f(x)$ and $g(x)$. Since $g(x) = 2^x$ passes through the points $(1, 2)$ and $(0, 1)$, we know the graph of $f(x)$ must pass through the points that are 1 unit to the left and 2 units down from these, namely $(0, 0)$ and $(-1, -1)$, respectively.



4. On the axes below, sketch a graph of $f(x) = \log_2(x + 1) - 3$.
Label at least two points on the curve.

The graph is shown along with the graph of $\log_2 x$ for reference.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Find the derivative of the function $f(x) = \dfrac{\sqrt{\cos x}}{e^{x^4+1}}$.

Using the quotient rule (and the chain rule) we have

$$f(x) = \frac{e^{x^4+1} \cdot \frac{1}{2}(\cos x)^{-1/2}(-\sin x) - \sqrt{\cos x} \cdot e^{x^4+1} \cdot 4x^3}{(e^{x^4+1})^2}$$

$$= \boxed{\frac{-\frac{e^{x^4+1}\sin x}{2\sqrt{\cos x}} - 4x^3 e^{x^4+1}\sqrt{\cos x}}{e^{2(x^4+1)}}}$$

2. Find the inverse of the function $f(x) = \dfrac{3}{2x-5}$.

   We have

$$\begin{aligned} y &= \frac{3}{2x-5} \\ 2x - 5 &= \frac{3}{y} \\ 2x &= \frac{3}{y} + 5 \\ x &= \frac{\frac{3}{y}+5}{2} \end{aligned}$$

   Therefore $\boxed{f^{-1}(x) = \dfrac{\frac{3}{x}+5}{2}}$

3. Simplify the expression $\log_3\left(\dfrac{27x^4}{3^{y+2}}\right)$.

   We have $\log_3\left(\dfrac{27x^4}{3^{y+2}}\right) = \log_3(27) + \log_3\left(x^4\right) - \log_3\left(3^{y+2}\right) = 3 + 4\log_3 x - (y+2) = \boxed{4\log_3 x - y + 1}$

4. Solve for x: $3\ln(x+2) = 6\ln x$.

   We can get rid of the logarithms on both sides by applying e to both sides, but first we must get rid of the coefficients. Dividing both sides by 3 we have

$$\begin{aligned} \ln(x+2) &= 2\ln x \\ \ln(x+2) &= \ln(x^2) \qquad \text{(this is a logarithm law)} \\ e^{\ln(x+2)} &= e^{\ln(x^2)} \\ x + 2 &= x^2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2, &\quad x = -1 \end{aligned}$$

   One of these, however, is not actually a solution to the original equation: $x = -1$ cannot be plugged into $\ln x$ on the right side of the equation. So the only solution is $\boxed{x = 2}$

5. A population of a certain strain of bacteria doubles every 24 hours. How long does it take for the population to triple?

   If t is measured in hours, then the population of bacteria at time t is $P(t) = P_0 \cdot 2^{t/24}$, where P_0 is the initial population. If the initial population triples, then $P(t) = 3P_0$. Therefore we have $3P_0 = P_0 \cdot 2^{t/24}$, or $3 = 2^{t/24}$. Solving for t, we get $\log_2(3) = \dfrac{t}{24}$, so $t = \boxed{24\log_2(3)\ (\approx 38)\text{ hours}}$

6. The half-life of ${}^{237}U$ (an isotope of uranium) has a half-life of 6.75 days. How long does it take for a 100g sample to decay to 10g?

The amount of ^{237}U after t days is $A(t) = 100 \cdot \left(\frac{1}{2}\right)^{t/6.75}$. We set $A(t) = 10$ and solve for t; we have

$$10 = 100 \cdot \left(\frac{1}{2}\right)^{t/6.75}$$
$$0.1 = \left(\frac{1}{2}\right)^{t/6.75}$$
$$\log_{1/2}(0.1) = \frac{t}{6.75}$$
$$\boxed{t = 6.75 \log_{1/2}(0.1)\ (\approx 22.4)\ \text{days}}$$