Math 75B Practice Final Solutions §§3.6-5.5 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. $\int (\sin x + 2\cos x)\, dx =$

   (a) $\boxed{-\cos x + 2\sin x + C}$

   (b) $-\cos x - 2\sin x + C$

   (c) $\cos x + 2\sin x + C$

   (d) $\cos x - 2\sin x$

   (e) $-\cos x - 2\sin x$

   This is straight out of the formulas.

2. If $\sin y + xy = 2x$, then $\dfrac{dy}{dx} =$

   (a) $\cos 2y$

   (b) $\boxed{\dfrac{2-y}{\cos y + x}}$

   (c) $\dfrac{\cos y}{2xy}$

   (d) $\dfrac{2x+y}{\cos y}$

   (e) $y + 2x\cos y$

   Differentiating both sides with respect to x and treating y as an implicit function of x, we have $\cos y \cdot \dfrac{dy}{dx} + x\dfrac{dy}{dx} + y = 2$. Solving for $\dfrac{dy}{dx}$, we get

$$\begin{aligned} \cos y \cdot \frac{dy}{dx} + x\frac{dy}{dx} &= 2 - y \\ (\cos y + x)\frac{dy}{dx} &= 2 - y \\ \frac{dy}{dx} &= \frac{2-y}{\cos y + x} \end{aligned}$$

3. If $3x\dfrac{dy}{dx} - 4 = \cos y \cdot \dfrac{dy}{dx}$, then $\dfrac{dy}{dx} =$

   (a) $\dfrac{3x}{4-\cos y}$

   (b) $\dfrac{3x-4}{\cos y}$

   (c) $\dfrac{4+\cos y}{3x}$

   (d) $\dfrac{\cos y}{3x-4}$

   (e) $\boxed{\dfrac{4}{3x-\cos y}}$

Here we already have an equation involving $\frac{dy}{dx}$, so it remains only to solve for it. We have

$$\begin{aligned} 3x\frac{dy}{dx} - \cos y\frac{dy}{dx} &= 4 \\ (3x - \cos y)\frac{dy}{dx} &= 4 \\ \frac{dy}{dx} &= \frac{4}{3x - \cos y} \end{aligned}$$

4. If $f(x) = \ln(7x^3 + 1)$, then $f'(x) =$

   (a) $\frac{1}{7x^3 + 1}$

   (b) $\frac{21x^2}{7x^3 + 1}$

   (c) $3\ln(7x + 1)$

   (d) $\frac{1}{21x^2}$

   (e) $\frac{7}{7x^3 - 21x^2}$

   Using the chain rule we have $f'(x) = \frac{1}{\text{“chunk”}} \cdot$ (derivative of “chunk”), i.e. $f'(x) = \frac{1}{7x^3 + 1} \cdot 21x^2 = \frac{21x^2}{7x^3 + 1}$.

5. Suppose you know that $f'(x) = g(x)$. Which of the following must be true?

   (a) $\int g(x)\,dx = f(x)$

   (b) $\boxed{\int g(x)\,dx = f(x) + C}$

   (c) $\frac{d}{dx}(g(x)) = f(x)$

   (d) $\frac{d}{dx}(g(x)) = f(x) + C$

   (e) All of the above are true.

   Antidifferentiation is the opposite of differentiation.

6. If $y = \int_0^{x^2} \tan t\,dt$, then $y' =$

   (a) $\boxed{2x\tan(x^2)}$

   (b) $\tan(x^2)$

   (c) $\tan x$

   (d) $2x\sec^2(x^2)$

   (e) $\sec^2(x^2)$

   This is F.T.C. I, combined with the chain rule. The $2x$ is the derivative of the “chunk” x^2.

7. The inflection point(s) of the function $y = 3x^5 - 5x^4 + 60x - 60$ is/are

(a) $(0, -60)$ only

(b) $(-1, -128)$ only

(c) $(-1, -128)$ and $(1, -2)$ only

(d) $\boxed{(1, -2) \text{ only}}$

(e) $(0, -60)$, $(1, -2)$, and $(-1, -128)$ only

To get inflection points we take the second derivative and set it equal to 0:

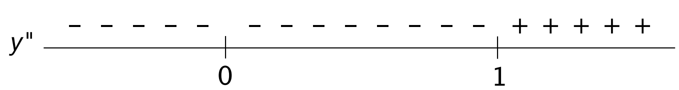
$$\begin{aligned} y' &= 15x^4 - 20x^3 + 60 \\ y'' &= 60x^3 - 60x^2 \overset{\text{set}}{=} 0 \\ 60x^2&(x-1) = 0 \\ x = 0 \quad &, \quad x = 1. \end{aligned}$$

But wait! There are no answer choices that have $x = 0$ and $x = 1$ and nothing else. What gives?

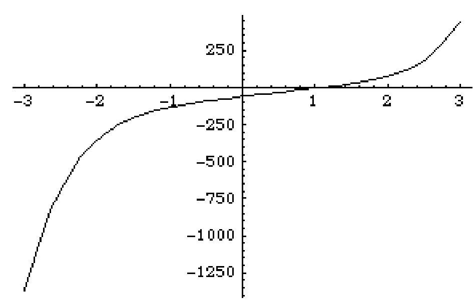
This is a trick question, since **the concavity must change** for a point to be considered an inflection point. So we must test the concavity on either side of 0 and 1.

$$\begin{aligned} y''(-1) &= -60 - 60 < 0 \\ y''\left(\tfrac{1}{2}\right) &= \frac{60}{8} - \frac{60}{4} < 0 \\ y''(2) &= 60 \cdot 8 - 60 \cdot 4 > 0 \end{aligned}$$

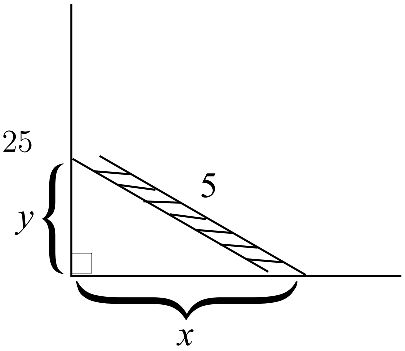
These test points give the results shown below.



Looking at the number line we see that the graph is concave down on both sides of 0, i.e. the concavity does not change. So $(0, -60)$ is not an inflection point. Here's what the graph looks like:



8. A ladder 5 meters long, leaning against a wall, begins to slide. According to the diagram at right,



(a) $\dfrac{dx}{dt} = -\dfrac{dy}{dt}$

(b) $2x\dfrac{dy}{dx} + 2y\dfrac{dx}{dy} = 0$

(c) $\boxed{x\dfrac{dx}{dt} = -y\dfrac{dy}{dt}}$

(d) $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 25$

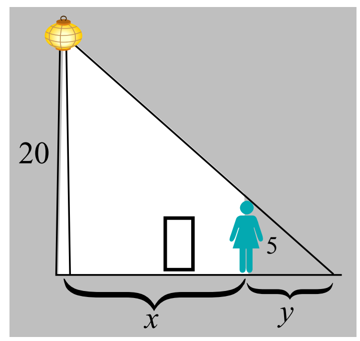
(e) $2x + 2y = 0$

This is a falling ladder problem. The setup, using the variables given in the diagram, is

$$x^2 + y^2 = 25.$$

Therefore the derivative of this equation with respect to t is $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$. We can rewrite this equation as $2x\dfrac{dx}{dt} = -2y\dfrac{dy}{dt}$ and then cancel the 2's to get $x\dfrac{dx}{dt} = -y\dfrac{dy}{dt}$.

9. A woman 5 ft. tall is walking toward a streetlight 20 ft. tall. According to the diagram at right, $\dfrac{dy}{dt}$ is



(a) the length of her shadow, a positive number for all t

(b) the rate at which the length of her shadow is changing, a positive number for all t

(c) $\boxed{\text{the rate at which the length of her shadow is changing, a negative number for all } t}$

(d) her speed, a positive number for all t

(e) her speed, a negative number for all t

Since y represents the length of her shadow, $\dfrac{dy}{dt}$ represents the rate of change of the length of her shadow. So the answer is either (**??**) or (9c).

This is a tricky question. On first inspection it may seem as though y is increasing, since the woman is walking away from her shadow. But as she gets closer to the lamp, her shadow gets shorter. Therefore the answer is (9c).

10. Which of the following is the linear approximation of the function $f(x) = \sqrt[3]{x}$ near the number $a = 1$?

(a) $y = \dfrac{1}{3}x + 1$

(b) $\boxed{y = \dfrac{1}{3}x + \dfrac{2}{3}}$

(c) $y = x - \dfrac{2}{3}$

(d) $y = x + 3$

(e) $y = 3x + 2$

This question is just asking for the equation of the tangent line to the graph of $f(x)$ at $x = 1$. So we need the slope and a point on the line. We have $f'(x) = \frac{1}{3}x^{-2/3}$, so the slope at $x = 1$ is $f'(1) = \frac{1}{3}$. Now the point of tangency is $(1, 1)$ since $f(1) = \sqrt[3]{1} = 1$. So using $y = mx + b$ we get

$$1 = \frac{1}{3} \cdot (1) + b$$
$$b = 1 - \frac{1}{3} = \frac{2}{3}.$$

Therefore the equation of the line is $\boxed{y = \frac{1}{3}x + \frac{2}{3}}$

11. $\int_0^4 |x - 3|\, dx =$

(a) 24

(b) 2

(c) 4

(d) 20

(e) $\boxed{5}$

This is similar to a problem from the last midterm. Use areas.

Fill-In.

1. The vertical asymptote(s) for the function $f(x) = \frac{x}{x^2 - 1}$ is/are $\underline{x = 1,\ x = -1}$ and the horizontal asymptote(s) is/are $\underline{y = 0}$.

The denominator is 0 for $x = 1$ and $x = -1$. Since the numerator is not also 0 for these x-values, there is a vertical asymptote at each of these places.

Since the degree of the bottom is bigger than the degree of the top, we have

$$\lim_{x \to \pm\infty} \frac{x}{x^2 - 1} = 0.$$

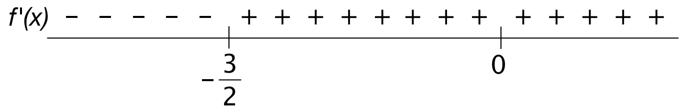
Therefore there is a horizontal asymptote at $y = 0$.

2. The graph of the function $f(x) = x^4 + 2x^3$ is increasing on the interval(s) $\underline{\left(-\frac{3}{2}, 0\right),\ (0, \infty)}$.

To check for increasing/decreasing we take the first derivative: $f'(x) = 4x^3 + 6x^2$. First, set it equal to 0 to find the critical numbers:

$$4x^3 + 6x^2 \overset{\text{set}}{=} 0$$
$$2x^2(2x + 3) = 0$$
$$x = 0 \quad , \quad x = -\frac{3}{2}.$$

Since the domain of $f'(x)$ is all real numbers, there are no "weird" critical numbers (numbers in the domain of $f(x)$ but not in the domain of $f'(x)$). So we set up a number line and check in between the above x-values:



Looking at the number line we see that the graph is increasing on the intervals $\left(-\frac{3}{2}, 0\right)$ and $(0, \infty)$.

3. According to Rolle's Theorem, the maximum number of real roots of the function $f(x) = 4x^5 + 2x - 3$ is $\underline{\ 1\ }$.

   According to Rolle's Theorem there is at most one more root than the number of solutions to the equation $f'(x) = 0$. We have $f'(x) = 20x^4 + 2 \stackrel{\text{set}}{=} 0 \Rightarrow$ no solutions! So there is at most 1 real root.

4. Given the initial guess $x_1 = 2$, the second approximation to a root of $g(x) = x^3 - 4x - 1$ using Newton's Method is $x_2 = \underline{\ \frac{17}{8}\ }$.

   We have $g'(x) = 3x^2 - 4$, so $x_2 = 2 - \frac{g(2)}{g'(2)} = 2 - \frac{2^3 - 4 \cdot 2 - 1}{3 \cdot 2^2 - 4} = 2 - \frac{-1}{8} = \frac{17}{8}$.

Graphs. *More accuracy = more points!*

1. For the function $f(x) = \frac{1}{3}x^3 - 2x$,

   (a) find the critical **points** and intervals of increase/decrease

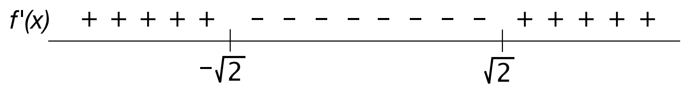
   We have

$$f'(x) = x^2 - 2 \stackrel{\text{set}}{=} 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

   The domain of $f'(x)$ is all real numbers, so $\sqrt{2}$ and $-\sqrt{2}$ are the only critical numbers. $f(\sqrt{2}) = \frac{1}{3}(\sqrt{2})^3 - 2\sqrt{2} = 2\sqrt{2}\left(\frac{1}{3} - 1\right) = -\frac{4\sqrt{2}}{3}$. $f(x)$ is an odd function (see part (c)), so we know that $f(-\sqrt{2}) = \frac{4\sqrt{2}}{3}$. Therefore the critical points are

$$\left(\sqrt{2}, -\frac{4\sqrt{2}}{3}\right), \left(-\sqrt{2}, \frac{4\sqrt{2}}{3}\right).$$

Now we set up a number line to find the intervals of increase and decrease:



$f(x)$ is increasing on $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$ and decreasing on $(-\sqrt{2}, \sqrt{2})$.

(b) find the inflection **points** and intervals of concave up/concave down

We repeat the process above for $f''(x)$: We have

$$f''(x) = 2x \overset{\text{set}}{=} 0 \Rightarrow x = 0.$$

$f(0) = 0$, so $(0,0)$ is a potential inflection point.
Since $f''(-1) = -2 < 0$, $f(x)$ is concave down for $x < 0$. Since $f(x)$ is an odd function (see part (c)), we know $f(x)$ is concave up for $x > 0$. Therefore $(0,0)$ is an inflection point, and $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

(c) discuss any symmetry $f(x)$ may or may not have

$f(-x) = \dfrac{1}{3}(-x)^3 - 2(-x) = -\dfrac{1}{3}x^3 + 2x = -\left(\dfrac{1}{3}x^3 - 2x\right) = -f(x)$, so $f(x)$ is an **odd function** (symmetric about the origin).

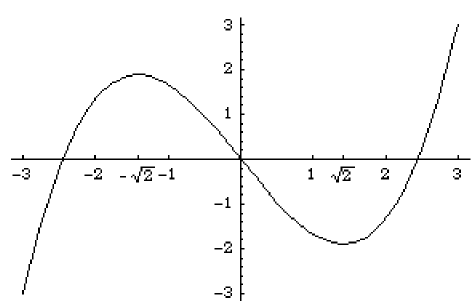
(d) find the equations of any vertical and/or horizontal asymptotes

There are no vertical or horizontal asymptotes since $f(x)$ is a polynomial.

(e) find the y-intercept

$f(0) = 0$, so the y-intercept is $(0,0)$.

(f) On the axes at right, sketch an accurate graph of $f(x)$.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Find the slope of the tangent line to the curve $x^2 + xy + y^3 = 7$ at the point $(2, 1)$.

   Using implicit differentiation and solving for $\frac{dy}{dx}$, we get

$$\begin{array}{ccccccc} x^2 & + & xy & + & y^3 & = & 7 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 2x & + & [x \cdot \frac{dy}{dx} + y] & + & [3y^2 \cdot \frac{dy}{dx}] & = & 0 \end{array}$$

$$x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = -2x - y$$

2. Use logarithmic differentiation to find the derivative of the function $f(x) = x^{\ln x}$.

   To use logarithmic differentiation we take the natural log of both sides and then use logarithm laws and implicit differentiation to help us get the derivative. We have

$$\begin{aligned} \ln(f(x)) &= \ln(x^{\ln x}) \\ \ln(f(x)) &= (\ln x) \ln x \qquad \text{(logarithm law } \ln(a^b) = b \ln a) \\ \ln(f(x)) &= (\ln x)^2 \\ \frac{f'(x)}{f(x)} &= 2 \ln x \cdot \frac{1}{x} \qquad \text{(implicit differentiation)} \\ \frac{f'(x)}{f(x)} &= \frac{2 \ln x}{x} \\ f'(x) &= \boxed{\frac{2 \ln x}{x} \cdot x^{\ln x}} \end{aligned}$$

   If you are feeling really fancy you can simplify the answer as $2 \ln x \cdot x^{\ln x - 1}$, but it is not required.

3. The area of a circular oil spill is increasing at the constant rate of 50π m^2 per minute. How fast is the radius of the spill increasing when the radius is 5 m?

   Simplify your answer and give units (e.g. grams, miles per gallon, etc.). You may use the fact that the area of a circle of radius r is $A = \pi r^2$.

   This is a related rates problem which concerns the quantities of radius and area of a circle. The equation $A = \pi r^2$ already is given which tells us the relationship between the two quantities, so to find out how their rates of change are related we take the derivative of this equation with respect to t:

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$

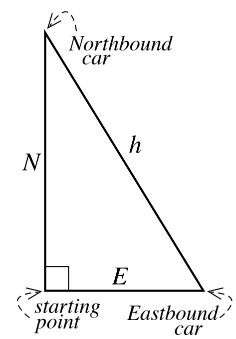
Now we plug in what we know and solve for what we don't. We are given that $\dfrac{dA}{dt} = 50$ and $r = 5$, and we want to find $\dfrac{dr}{dt}$. We have

$$50 = 2\pi \cdot 5 \cdot \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{5}{\pi}\text{m/min.}}$$

4. Two cars start moving from the same point. One travels north at 40 mi/h and the other travels east at 30 mi/h. At what rate is the distance between the cars increasing two hours later? *Simplify your answer and give units (e.g. feet, kilograms, etc.).*

The situation is pictured at right. Since the rates of change of the distance of the two cars from the starting point, and of the two cars from each other, are mentioned in the problem, we need to label all of these distances.



The equation for step 1 is

$$N^2 + E^2 = h^2. \tag{1}$$

For step 2 we take the derivative with respect to t:

$$2N\frac{dN}{dt} + 2E\frac{dE}{dt} = 2h\frac{dh}{dt} \tag{2}$$

Finally, for step 3 we plug in the information given: after 2 hours the northbound car has gone 80 miles (40 mi./hr.* 2 hrs.) and the eastbound car has gone 60 miles. So $N = 80$ and $E = 60$. Now we can use equation (1) to get h. We get $h = \sqrt{80^2 + 60^2} = \sqrt{10000} = 100$ miles. We have

$$N = 80 \qquad \frac{dN}{dt} = 40$$

$$E = 60 \qquad \frac{dE}{dt} = 30$$

$$h = 100 \qquad \frac{dh}{dt} = \text{ what we want}$$

So we plug in these values to equation (2) and solve for $\frac{dh}{dt}$:

$$
\begin{aligned}
2\cdot 80\cdot 40 + 2\cdot 60\cdot 30 &= 2\cdot 100\cdot \frac{dh}{dt}\\
\cancel{80}_{20}\cdot 40 + \cancel{60}_{15}\cdot 30 &= \cancel{100}_{25}\frac{dh}{dt}\\
800 + 450 &= 25\frac{dh}{dt}\\
\frac{dh}{dt} = \frac{1250}{25} &= \boxed{50 \text{ mi./hr.}}
\end{aligned}
$$

5. A farmer has 400 meters of fencing with which to fence 3 sides of a rectangular horse corral. What is the maximum area she can enclose?

   The objective of this problem is to **maximize the area**. Let x be the width and y the length of the corral. A formula for the area, then, is $A = xy$. We know $2x + y = 400$, so $y = 400 - 2x$. Therefore the area, in terms of x, is

$$A(x) = x(400 - 2x) = 400x - 2x^2.$$

   Now we find where the absolute maximum of the area function is:

$$
\begin{aligned}
A'(x) = 400 - 4x &\overset{\text{set}}{=} 0\\
4x &= 400\\
x &= 100
\end{aligned}
$$

   The area is maximized at $x = 100$. The problem asks for the maximum area, so we plug in 100: $A(100) = 100(400 - 2\cdot 100) = 100(200) = \boxed{20{,}000 \text{ m}^2}$

6. Evaluate $\int_{-1}^{2} (x^2 + 2)\, dx$.

   We have

$$
\begin{aligned}
\int_{-1}^{2} (x^2 + 2)\, dx &= \frac{1}{3}x^3 + 2x \Big|_{-1}^{2}\\
&= \left(\frac{1}{3}\cdot 2^3 + 2\cdot 2\right) - \left(\frac{1}{3}(-1)^3 + 2(-1)\right)\\
&= \frac{8}{3} + 4 + \frac{1}{3} + 2 = 3 + 4 + 2 = \boxed{9}
\end{aligned}
$$

7. Evaluate $\displaystyle\int x(3x^2+1)^5\,dx$.

Let $u = 3x^2+1$. Then $du = 6x\,dx$. Futzing the 6, we get

$$\begin{aligned}\int x(3x^2+1)^5\,dx &= \frac{1}{6}\int 6x(3x^2+1)^5\,dx \\ &= \frac{1}{6}\int u^5\,du \\ &= \frac{1}{6}\cdot\frac{1}{6}u^6 + C \\ &= \boxed{\frac{1}{36}(3x^2+1)^6 + C}\end{aligned}$$

8. Evaluate $\displaystyle\int_0^1 x\cos(x^2+1)\,dx$.

Let $u = x^2+1$. Then $du = 2x\,dx$. Also the new limits become

$$\begin{aligned}1:&\quad u = 1^2+1 = 2 \\ 0:&\quad u = 0^2+1 = 1.\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_0^1 x\cos(x^2+1)\,dx &= \frac{1}{2}\int_0^1 2x\cos(x^2+1)\,dx \\ &= \frac{1}{2}\int_1^2 \cos u\,du = \frac{1}{2}\sin u\,\Big|_1^2 \\ &= \boxed{\frac{1}{2}(\sin(2)-\sin(1))}\end{aligned}$$